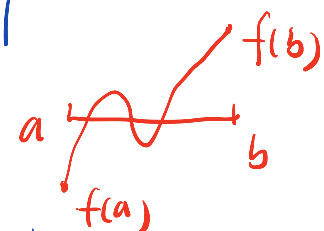
Chapter 12 Intermediate Value Theorem

Def c is called a zero of $f(x)$ if $f(c)=0$

Def We say $f(x)$ changes sign on $[a,b]$ if $f(x)$ is defined on $[a,b]$ and $f(x)$ has opposite sign at a & b:

$$\left.\begin{array}{ll} \text{either} & f(a)>0,\ f(b)<0 \\ \text{or} & f(a)<0,\ f(b)>0 \end{array}\right\} \Longleftrightarrow f(a)f(b)<0$$



Bolzano's Theorem

$$\left.\begin{array}{l} f(x) \text{ is continuous on } [a,b] \\ f(a)f(b)<0 \end{array}\right\} \Rightarrow f(x) \text{ has a zero on } [a,b]$$

Proof: WLOG (Without loss of generality), we assume $f(a)<0$ and $f(b)>0$

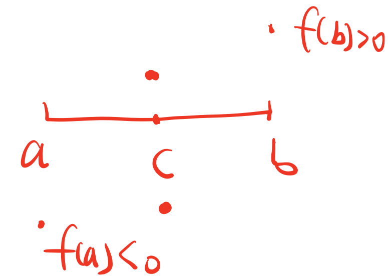
(because the other case will be similar).

Want to show

$$f(a)<0,\ f(b)>0 \Rightarrow \exists\, c \in (a,b),\ f(c)=0$$

Use bisection to generate nested intervals such that $f(a_n)<0$, $f(b_n)>0$

Let $a_0 = a$, $b_0 = b$, $C_0 = \frac{a+b}{2}$



1) If $f(C_0) = 0$, proof is completed

2) If $f(C_0) > 0$, pick left half, call it $[a_1, b_1]$

3) If $f(C_0) < 0$, pick right half, call it $[a_1, b_1]$

Continue this process, then

either proof is completed (we find a zero)

or we get nested intervals

$$[a_1, b_1] \supset [a_2, b_2] \supset \cdots \supset [a_n, b_n] \supset \cdots$$

Satisfying ① length is halved thus going to 0.

② $f(a_n) < 0$, $f(b_n) > 0$

The length of $[a_n, b_n]$ is $\frac{b-a}{2^n} = \frac{1}{2^n}(b-a) \to 0$

Nested interval Thm $\Rightarrow \lim_{n\to\infty} a_n = c = \lim_{n\to\infty} b_n$

$f(a_n) < 0 \Rightarrow \lim_{n\to\infty} f(a_n) \leq 0$

Sequence Limit Location Thm

$a_n \leq M \Rightarrow \lim_{n\to\infty} a_n \leq M$.

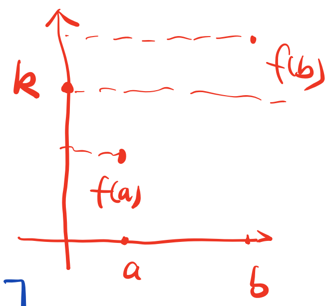
$f(b_n) > 0 \Rightarrow \lim_{n\to\infty} f(b_n) \geq 0$

$f(x)$ is continuous on $[a, b]$

$\Rightarrow f(x)$ is continuous at $c \in [a, b]$

which Thm? $\Rightarrow \begin{cases} \lim\limits_{n\to\infty} f(a_n) = f(c) \Rightarrow f(c) \leq 0 \\ \lim\limits_{n\to\infty} f(b_n) = f(c) \Rightarrow f(c) \geq 0 \end{cases}$

$\Rightarrow \quad f(c) = 0.$



<u>Intermediate Value Theorem</u>

If $f(x)$ is continuous on $[a, b]$,

then for any value k between $f(a)$ and $f(b)$,

there is some $c \in [a, b]$ s.t. $f(c) = k$.

Proof: WLOG, assume $f(a) < f(b)$

1) If $k = f(a)$, $c = a$
2) If $k = f(b)$, $c = b$
3) If $f(a) < k < f(b)$, then consider

$$g(x) = f(x) - k$$

$$g(a) = f(a) - k < 0$$
$$g(b) = f(b) - k > 0$$

Bolzano's Thm $\Rightarrow \exists c \in (a, b)$ s.t. $g(c) = 0$

$$\Rightarrow f(c) = k.$$

HW # 9

P1. Assume $\begin{cases} f(x) \text{ is continuous for all } x \in \mathbb{R} \\ f(a+b) = f(a) + f(b) \text{ for all } a, b \end{cases}$

Prove $f(x) = Cx$ where $C = f(1)$ as follows

① Prove $f(x) = Cx$ for $x = n, \frac{1}{n}, \frac{m}{n}$, m, n are integers

② Use continuity to show it's true for all x.

Hint: Let c be any irrational number

C_n be a real number of finite decimals

with same integer part and first n decimals

Then $C_n \to C$ because $|C_n - C| < 10^{-n}$.

$C = 3.1415926\ldots$

$C_1 = 3.1$

$C_2 = 3.14$

$C_3 = 3.141$

Proof: ① If $n > 0$

$$f(n) = f(n-1+1) = f(n-1) + f(1)$$

$$= f(n-2+1) + f(1)$$

$$= f(n-2) + f(1) + f(1)$$

$$= \cdots = f(1) + \cdots + f(1) = n \cdot f(1)$$

If $n=0$

$a=0,\ b=0 \Rightarrow f(0+0)=f(0)+f(0)$

$\Rightarrow f(0)=0$

If $n>0$, $f(n+(-n))=f(n)+f(-n)$

$\Rightarrow f(0)=n\cdot f(1)+f(-n)$

$\Rightarrow f(-n)=-n\cdot f(1)$

$f(\frac{1}{2}+\frac{1}{2})=f(\frac{1}{2})+f(\frac{1}{2}) \Rightarrow 2f(\frac{1}{2})=f(1)$

② Let x be irrational

x_n be a real number of finite decimals

with same integer part and first n decimals

Then $x_n \rightarrow x$

$f(x_n)=x_n\cdot f(1)$

Want to show $f(x)=x\cdot f(1)$

P2 "Ruler function"

$$f(x) = \begin{cases} \frac{1}{2^n}, & \text{if } x = \frac{b}{2^n} \text{ for some positive integer } n \text{ some odd integer } b \\ 0, & \text{otherwise} \end{cases}$$

(a) Prove $f(x)$ is discontinuous at points $\frac{b}{2^n}$ (odd b)

(b) Prove $f(x)$ is continuous at other points.

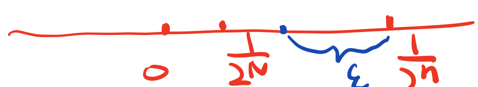
Idea for (a): $\exists \varepsilon > 0, \forall \delta, \exists x \in (x_0-\delta, x_0+\delta), |f(x) - f(x_0)| \geq \varepsilon$

want to find some x that is very close to x_0 but $|f(x) - f(x_0)| \geq \varepsilon$

(a) $x_0 = \frac{b}{2^n}$, $x = \frac{1}{2^N} + x_0 = \frac{1}{2^N} + \frac{b}{2^n} = \frac{1 + b\cdot 2^{N-n}}{2^N}$

Then $|x - x_0| = \frac{1}{2^N} < \delta$, $\forall \delta > 0$ if $N \gg 1$.

$|f(x) - f(x_0)| = \left|\frac{1}{2^N} - \frac{1}{2^n}\right| \geq \frac{1}{2}\frac{1}{2^n}$, $N \gg 1$



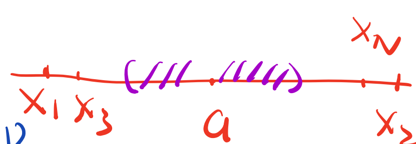
Idea for (b): $\forall \varepsilon > 0, \exists \delta, \forall x \in (a-\delta, a+\delta), |f(x) - f(a)| < \varepsilon$.

For fixed a satisfying $f(a) = 0$,

Let x_n be the nearest number to a s.t. $f(x_n) = \frac{1}{2^n}$

let δ be the minimum of the set

$\{|a - x_1|, |a - x_2|, \cdots, |a - x_N|\}$



$\forall x \in (a-\delta, a+\delta)$, what do we know about $f(x)$?

$|f(x) - f(a)| \leq \frac{1}{2^N} < \varepsilon$ if $N \gg 1$.

?

$\forall x \in (a-\delta, a+\delta)$, $f(x) \neq \frac{1}{2}$

because $f(x_1) = \frac{1}{2}$ and x_1 is the nearest point to a and satisfies $f(x_1) = \frac{1}{2}$

Similarly, $f(x) \neq \frac{1}{4}, \frac{1}{8}, \ldots, \frac{1}{2^N}$

So $f(x)$ can be $0, \frac{1}{2^{N+1}}, \frac{1}{2^{N+2}}, \ldots$

Theorem 11.5 B

If for any $\{x_n\}$ satisfying $x_n \to a$ we have $f(x_n) \to L$

then $\lim_{x \to a} f(x) = L$.

$\boxed{P3}$ Prove Theorem 11.5 B by contraposition:

prove that

$$\lim_{x \to a} f(x) \neq L \Rightarrow \exists \{x_n\} \text{ s.t. } x_n \to a$$
$$\text{but } \lim_{n \to \infty} f(x_n) \neq L$$

Proof: $\lim_{x \to a} f(x) = L$: $\forall \varepsilon > 0$, $\exists \delta > 0$, $\forall \underset{x \neq a}{x \in (a-\delta, a+\delta)}$,
$|f(x) - L| < \varepsilon$

$\lim_{x \to a} f(x) \neq L$: $\exists \varepsilon > 0$, $\forall \delta > 0$, $\exists \underset{x \neq a}{x \in (a-\delta, a+\delta)}$,
$|f(x) - L| \geq \varepsilon$.

$\exists \varepsilon > 0$, for $\delta = \frac{1}{3^n}$, $\exists x_n \in (a - \delta, a + \delta)$

$$\begin{cases} |f(x_n) - L| \geq \varepsilon \\ |x_n - a| < \frac{1}{3^n} \Rightarrow x_n \to a \end{cases}$$

$\boxed{P4}$ Assume $\begin{cases} f(x) \text{ is continuous on } [a,b] \\ f([a,b]) = [f(a), f(b)] : f(a) \leq f(b) \\ \qquad \text{and } f(a) \leq f(x) \leq f(b),\ \forall x \in [a,b] \\ f(x) \text{ never repeats a value.} \end{cases}$

Prove $f(x)$ is strictly increasing

Hint: use intermediate value Thm in an indirect proof.

Proof: Assume not strictly increasing,

since no values are repeated,

there are $a_1 < b_1$ s.t. $f(a_1) > f(b_1)$