Review 1

Completeness Thm

A bounded monotone sequence converges.

B-W Thm

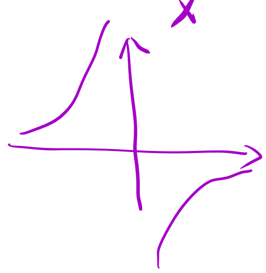
A bounded sequence has a convergent subsequence.

HW/Exam Problem

$f(x)$ is locally increasing on some interval then $f(x)$ is increasing on that interval.

False Statement: $f(x)$ is locally increasing on its domain then $f(x)$ is increasing.

Example: $f(x) = -\frac{1}{x}$ on $(-\infty, 0) \cup (0, +\infty)$



Global Properties always imply the local properties.

Question: when would the local one imply the global one?

<u>Theorem</u> $f(x)$ is locally bounded on $[a,b]$

$\Rightarrow f(x)$ is bounded on $[a,b]$

Proof: Assume $f(x)$ is not bounded on $[a,b]$.

bounded: $\exists M \geq 0$ s.t. $\forall x \in [a,b]$, $|f(x)| \leq M$.

$\forall M \geq 0$, $\exists x_M \in [a,b]$ s.t. $|f(x_M)| > M$.

$\Rightarrow$ For $M = n$, there exists $x_n \in [a,b]$

$a \leq x_n \leq b$

s.t. $|f(x_n)| > n$.

$\{x_n\} \subset [a,b]$, $|f(x_n)| > n$.

$a \leq x_n \leq b \Rightarrow \{x_n\}$ is bounded

(B-W Thm) $\Rightarrow$ Convergent Subsequence $\{x_{n_i}\}$

$|f(x_{n_i})| > n_i \geq i$

Limit Location Thm

$a \leq x_{n_i} \leq b$ $\Big\} \Rightarrow a \leq \lim\limits_{i \to \infty} x_{n_i} \leq b$

$\Rightarrow x_{n_i} \to L \in [a,b]$

Intuition for deriving contradiction:

$|f(x_{n_i})| > n_i \Rightarrow \lim_{i\to\infty} |f(x_{n_i})| = +\infty$

$f(x)$ is locally bounded on $[a,b]$

$\Rightarrow \forall x_0 \in [a,b]$, $\exists \delta > 0$ s.t.

$f(x)$ is bounded on $(x_0-\delta, x_0+\delta) \cap [a,b]$

$\Rightarrow$ For $L \in [a,b]$, $\exists \delta > 0$ s.t.

① $\exists M > 0$, $|f(x)| \le M$, $\forall x \in (L-\delta, L+\delta) \cap [a,b]$

$x_{n_i} \to L \Rightarrow |x_{n_i} - L| < \delta$, $i >> 1$

② $|f(x_{n_i})| > n_i \ge i > M$, $i >> 1$

Contradiction between ① and ②.

Chapter 13 Compact Intervals

Def A set $S \subseteq \mathbb{R}$ is called sequentially compact if every sequence in S has a convergent subsequence converging to $L \in S$.

Example: ① Theorem $[a,b]$ is sequentially compact

Proof: $\forall \{x_n\} \subset [a,b]$

$a \le x_n \le b$ $\Rightarrow$ There is a convergent subsequence x_{n_i}

B-W Thm

$$\left.\begin{array}{l} a \le x_{n_i} \le b \\ \text{Limit Location Thm} \end{array}\right\} \Rightarrow a \le \lim_{i\to\infty} x_{n_i} \le b$$

② $(a, b]$ is not sequentialy compact.

Proof: $x_n = a + \frac{b-a}{n} \to a$

$\Rightarrow$ Any subsequence of $\{x_n\}$ goes to $a \notin S = (a, b]$.

③ $[0, +\infty)$ is not sequentialy compact.

Proof: $x_n = n$

$\Rightarrow \{x_n\}$ has no convergent subsequence.

<u>Theorem</u> $\{a_n\}$ converges $\Leftrightarrow \{a_n\}$ is <u>Cauchy</u>

$(|a_m - a_n| < \varepsilon, m, n \gg 1)$

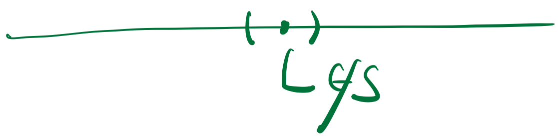
To show $\{x_{n_i}\}$ is not convergent:

$|x_{n_i} - x_{n_j}| \ge 1$ for any i, j

$\Rightarrow$ not Cauchy $\Rightarrow$ not convergent.

④ $S = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\} = \{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ is sequentially compact?

$\forall \{x_n\} \subset S$ $\quad x_{n_i} \to L$



$(L-\varepsilon, L+\varepsilon)$ does not overlap with S

<u>Theorem</u> $f(x)$ is continuous on a compact interval $I \Rightarrow f(x)$ is bounded on I.

Proof: First, we prove $f(x)$ is bounded above by contradiction.

Assume $f(x)$ is not bounded above.

$\forall M \geq 0$, $\exists x_M \in I$ s.t. $f(x_M) > M$.

$\Rightarrow$ For $M = n$, $\exists x_n \in I$ s.t. $f(x_n) > n$.

$$\left.\begin{array}{l} I \text{ is compact} \\ \{x_n\} \subset I \end{array}\right\} \Rightarrow \begin{cases} x_{n_i} \to L \in I \\ f(x_{n_i}) > n_i \end{cases}$$

$$\left.\begin{array}{l} \lim\limits_{i\to\infty} x_{n_i} = L \\ f(x) \text{ is cont.} \end{array}\right\} \Rightarrow \lim_{i\to\infty} f(x_{n_i}) = f(L)$$

$$\left.\begin{array}{l} f(x_{n_i}) > n_i \\ n_i \to +\infty \end{array}\right\} \Rightarrow \lim_{i\to\infty} f(x_{n_i}) = +\infty$$

$\Rightarrow f(L) = +\infty$, $L \in I$.

$\Rightarrow f(x)$ is not defined at $L \in I$.

$\Rightarrow$ Contradiction