Basic Logic (Appendix A in textbook)

- If-then statements:

  mathematical statements are often written as

  1) if A, then B
     hypothesis, conclusion

  2) $A \Rightarrow B$

Example: ① $f(x)$ is differentiable $\Rightarrow$ $f(x)$ is continuous

This statement is true.

Proof: $\lim_{\Delta x \to 0} \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$ exists $\Rightarrow \lim_{\Delta x \to 0}[f(x_0+\Delta x)-f(x_0)]=0$

$\Rightarrow \lim_{\Delta x \to 0} f(x_0+\Delta x)=f(x_0) \Rightarrow f(x)$ is cont. at x_0

② $f(x)$ is continuous $\Rightarrow$ $f(x)$ is bounded

This statement is false

Proof: A counterexample is $f(x)=x$.

- Converse statement:

  For $A \Rightarrow B$, its converse is $B \Rightarrow A$

- Equivalent statement: $A \Leftrightarrow B$ means $\begin{cases} A \Rightarrow B \\ B \Rightarrow A \end{cases}$

Alternative ways for "$\Leftrightarrow$"

same $\begin{cases} ① \ A \Leftrightarrow B \\ ② \ A \text{ if and only if } B \\ ③ \ A \text{ iff } B \\ ④ \ A \text{ is a necessary and sufficient condition of } B. \end{cases}$

same $\begin{cases} ① \ A \Rightarrow B \\ ② \ A \text{ is a sufficient condition of } B. \end{cases}$

same $\begin{cases} ① \ A \Leftarrow B \ (\text{or } B \Rightarrow A) \\ ② \ A \text{ is a necessary condition of } B. \end{cases}$

- Stronger and weaker:

  If $A \Rightarrow B$ is true but $B \Rightarrow A$ is false, we say

  A is stronger than B and B is weaker than A.

  Example: "$f(x)$ is differentiable" is stronger than "$f(x)$ is continuous".

- Negation: not A

  A is "$f(x)$ is continuous at any $x \in \mathbb{R}$".

  Negation of A is "$f(x)$ is discontinuous at some point on $\mathbb{R}$".

- Contrapositive form:

  The contraposition of "$A \Rightarrow B$" is

"Not $B \Rightarrow$ Not A"

Basic methods

I. Indirect Proof:

To prove "$A \Rightarrow B$", we can

① assume A is true, then show B is true.

or

② show "Not $B \Rightarrow$ Not A"

Assume B is not true, then show A is false

II. Proof by Contradiction:

To prove "$A \Rightarrow B$", we can

③ assume B is not true, derive a contradiction with A being true.

III. Mathematical Induction (Two Steps)

Example: Given $x \neq 1$, prove that $1+x+\cdots+x^n = \dfrac{1-x^{n+1}}{1-x}$, $\forall n \geq 1$ (*)

1) If $n=1$, LHS $= 1+x$

$$RHS = \frac{1-x^2}{1-x} = \frac{(1-x)(1+x)}{1-x} = 1+x$$

So $(*)$ is true for $n=1$.

2) Assume $(*)$ is true for $n=k$

Want to show $(*)$ is true for $n=k+1$.

We have $1+x+\cdots+x^k = \frac{1-x^{k+1}}{1-x}$

$$\Rightarrow 1+x+\cdots+x^k+x^{k+1} = \frac{1-x^{k+1}}{1-x}+x^{k+1}$$

$$RHS = \frac{1-x^{k+1}}{1-x}+x^{k+1} = \frac{1-x^{k+1}}{1-x} + \frac{x^{k+1}(1-x)}{1-x}$$

$$= \frac{1-x^{k+2}}{1-x}$$

$$\Rightarrow 1+x+\cdots+x^k+x^{k+1} = \frac{1-x^{k+2}}{1-x}$$

$\Rightarrow$ $(*)$ is true for $n=k+1$.

By (Math) Induction, $(*)$ is true for any $n \geq 1$.

To prove "$A \Rightarrow B$", we convert intuition to rigorous reasoning by

$$\left.\begin{array}{l}\text{Definitions}\\ \text{Theorems}\\ \text{Assumptions in A}\end{array}\right\} \xRightarrow{\text{Logic}} \text{rigorous proof.}$$

Chapter 2

$a \in \mathbb{R}$ means a is a real number

the set of all real numbers

- Inequalities

  a is positive if $a > 0$

  a is non-negative if $a \geq 0$

① $\left.\begin{array}{l} a<b \\ c<d \end{array}\right\} \Rightarrow a+c<b+d$

② $\left.\begin{array}{l} a<b \\ c<d \end{array}\right\} \not\Rightarrow ac<bd$

③ $\left.\begin{array}{l} a<b \\ c<d \\ a,b,c,d \geq 0 \end{array}\right\} \Rightarrow ac<bd$

④ $a<b \Rightarrow -a>-b$

$\left.\begin{array}{l} a<b \\ a,b>0 \end{array}\right\} \Rightarrow \frac{1}{a} > \frac{1}{b}$

Q: $\left.\begin{array}{l} a<b \\ a,b>0 \end{array}\right\} \overset{?}{\Rightarrow} \sqrt{a}<\sqrt{b}$

Proof : The contrapositive is " $\sqrt{a} \geq \sqrt{b} \Rightarrow a \geq b$ "

$\left.\begin{array}{l}\sqrt{a} \geq \sqrt{b} \\ \sqrt{a} \geq \sqrt{b}\end{array}\right\} \Rightarrow a \geq b$. $\square$

- Estimation :

Def If $K < c < M$, we say

K is a lower bound (or estimate) of c

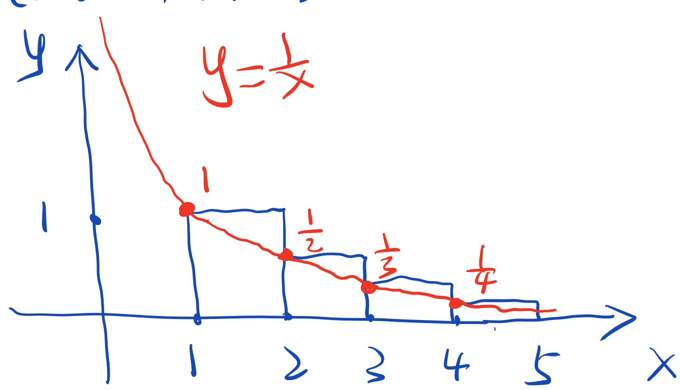
and M is an upper bound (or estimate) of c.

Example : $a_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$

Proposition $\{a_n\}$ is Harmonic Sum/Series

Show that $\{a_n\}$ is not bounded above.

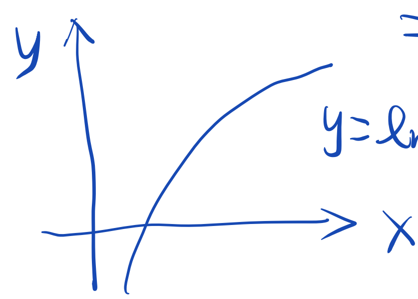
$\boxed{\text{Intuition}}$



Solution :

$$a_n > \int_1^{n+1} \frac{1}{x}\, dx = \ln x \Big|_1^{n+1}$$

$$= \ln(n+1)$$

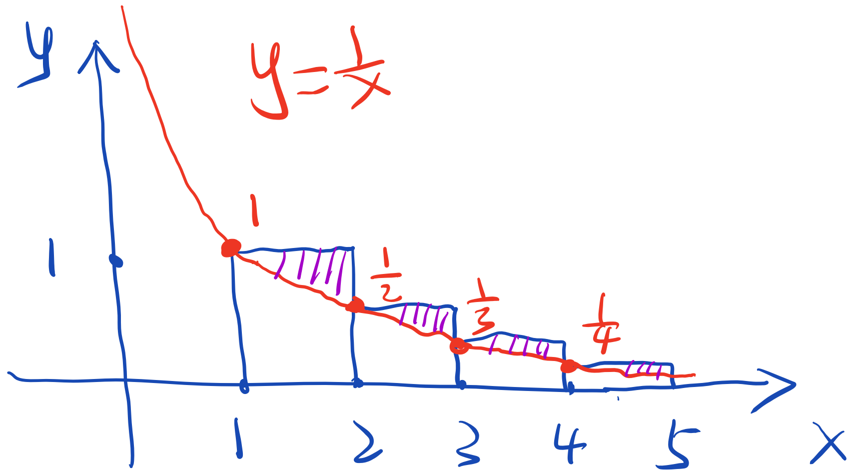


$\left.\begin{array}{l}\ln(x) \text{ is not bounded above} \\ a_n > \ln(n+1)\end{array}\right\} \Rightarrow \{a_n\}$ is not bounded above.

Example: $b_n = 1+\frac{1}{2}+\cdots+\frac{1}{n} - \ln(n+1)$, $n \geq 1$

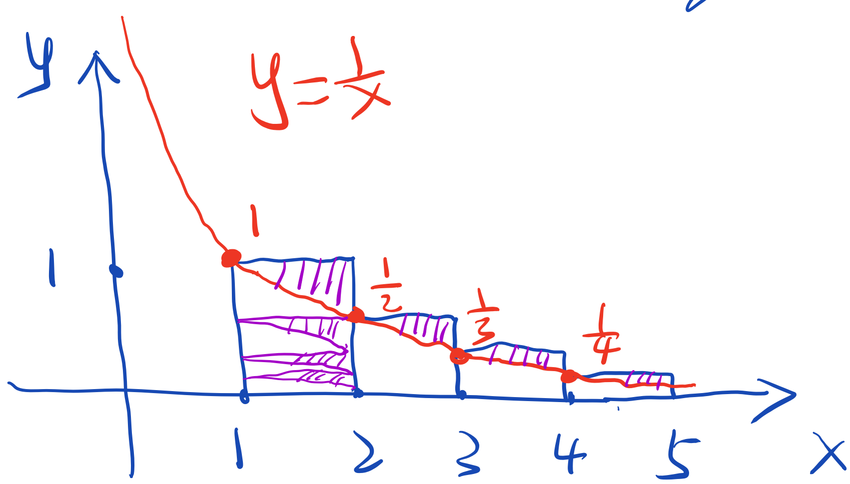
Proposition Show $\{b_n\}$ has a limit

Solution: Want to show $\{b_n\}$ is increasing and bounded above



$\int_1^{n+1} \frac{1}{x}dx = \ln(n+1) \Rightarrow b_n$ is the sum of purple area.

$\Rightarrow \{b_n\}$ is increasing



By the picture, $b_n < 1 \Rightarrow \{b_n\}$ is bounded above.

(By the Theorem) So $\{b_n\}$ has a limit.

Definition A number L is the limit of an increasing $\{a_n\}$ if, for any given integer k, all the a_n after some place in the sequence agree with L to k decimal places.

Theorem An increasing and bounded above sequence has a limit.

Definition A number L is the limit of a decreasing $\{a_n\}$ if, for any given integer k, all the a_n after some place in the sequence agree with L to k decimal places.

Theorem A decreasing and bounded below sequence has a limit.

Def $\{a_n\}$ is bounded if there are constants B and C s.t.

$$C \leq a_n \leq B, \quad \forall n$$

Def $\{a_n\}$ is monotone if it is increasing or decreasing.

Theorem (Completeness of Real Numbers)
A monotone bounded sequence of real numbers

has a limit (which is of course still a real number)

Remark: ① Rational numbers are incomplete in the sense that a monotone bounded sequence of rational numbers does not necessarily have a rational number limit.

② Remove all rational numbers from the real line, we still see a line.

(much more irrational numbers than rationals)

- Absolute value

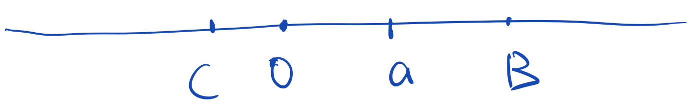
① $|a| = \begin{cases} a, & \text{if } a \geq 0 \\ -a, & \text{if } a < 0 \end{cases}$

② $|a-b|$ is the distance between a and b

③ $|a| \leq M \iff -M \leq a \leq M$

④ $C \leq a \leq B \implies |a| \leq M$

where $M = \max\{|C|, |B|\}$



Proof: $-M \leq -|C| \leq C \leq a \leq B \leq |B| \leq M$

⑤ $|a+b| \leq |a| + |b|$
$|a+b+c| \leq |a| + |b| + |c|$ } triangle inequality

Proof: $-|a| \leq a \leq |a|$

$-|b| \leq b \leq |b|$

⑥ $|a-b| \geq |a| - |b|$ Proof: $\underset{\overset{\|}{|a|}}{|(a-b)+b|} \leq |a-b| + |b|$

$|a+b| \geq |a| - |b|$ Proof: $|a+b+(-b)| \leq |a+b| + |-b|$

Ex: Find an upper estimate of

$$a_n = \sum_{i=1}^{n} \frac{1}{2^i} \cos(it), \text{ where } t \text{ is a variable.}$$

$$= \frac{1}{2}\cos t + \frac{1}{4}\cos(2t) + \cdots + \frac{1}{2^n}\cos(nt)$$

Sol: $$|a_n| = \left| \sum_{i=1}^{n} \frac{1}{2^i} \cos(it) \right|$$

$$\leq \sum_{i=1}^{n} \left| \frac{1}{2^i} \cos(it) \right| = \sum_{i=1}^{n} \frac{1}{2^i} |\cos(it)|$$

$$\leq \sum_{i=1}^{n} \frac{1}{2^i} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} = \boxed{?}$$

$$1 + x + \cdots + x^n = \frac{1 - x^{n+1}}{1-x}, \quad \forall n \geq 1$$

(*)