Uniform Continuity on some interval

① $f(x)$ is continuous on (a,b) :

δ depends on x_0, ε.

$\forall x_0 \in (a,b)$, $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.

$\forall x \in (x_0-\delta, x_0+\delta) \cap (a,b)$, $|f(x)-f(x_0)| < \varepsilon$.

② Uniform Continuity

$f(x)$ is uniformly continuous on (a,b) if

$\forall x_0 \in (a,b)$, $\forall \varepsilon > 0$, there is $\delta > 0$ which only depends on ε s.t.

$\forall x \in (x_0-\delta, x_0+\delta) \cap (a,b)$, $|f(x)-f(x_0)| < \varepsilon$.

③ Definition (Uniform Continuity)

$f(x)$ is uniformly continuous on an interval I if

$\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.

$\forall x, y \in I$ satisfying $|x-y| \leq \delta$, $|f(x)-f(y)| < \varepsilon$.

"$f(x)$ is not uniformly continuous on I" means

$\exists \varepsilon > 0$, s.t. $\forall \delta > 0$,

$\exists x, y \in I$ satisfying $|x-y| \leq \delta$, $|f(x)-f(y)| \geq \varepsilon$

Example: $f(x) = x^2$ is not uniformly continuous on $I = (0, +\infty)$

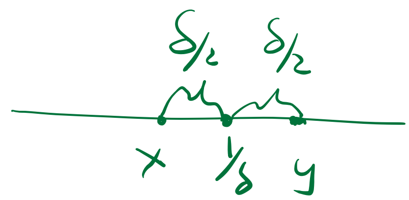
Intuition: $|f(x) - f(y)| = |x^2 - y^2| = |x-y| \cdot |x+y|$

$|x-y| = \delta$, $|x^2 - y^2| \geq \varepsilon$ if $|x+y| = \frac{\varepsilon}{\delta}$.

Proof: Pick $\varepsilon = 1$, $\forall \delta > 0$,

consider $x = \frac{1}{\delta} - \frac{\delta}{2}$

$y = \frac{1}{\delta} + \frac{\delta}{2}$



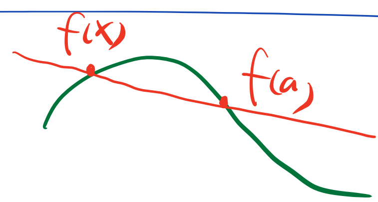
Then $|x - y| \leq \delta$, and $|x+y| = \frac{2}{\delta}$

$$|f(x) - f(y)| = |x^2 - y^2| = |x - y| \cdot |x+y| \leq \delta \cdot \frac{2}{\delta} = 2 > \varepsilon.$$

Theorem If I is a compact interval,

$f(x)$ is continuous on $I \Rightarrow f(x)$ is uniformly cont. on I.

Chapter 14 Differentiation



Def Assume $f(x)$ is defined for $x \approx a$

$\exists \delta > 0$, $\forall x \in (a - \delta, a + \delta)$

① We write $f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a} = \lim_{\Delta x \to 0} \frac{f(a+\Delta x)-f(a)}{\Delta x}$

if the limit exists.

② $f'(a)$ is called derivative of $f(x)$ at a

③ $f(x)$ is differentiable at a.

<u>Def</u> $f(x)$ is called differentiable on an open interval I if $f(x)$ is differentiable at any point of I.

<u>Def</u> $f(x)$ is differentiable on $[a,b]$ if

① $f(x)$ is differentiable on (a,b)

② $f'(a+) = \lim_{x \to a+} \frac{f(x)-f(a)}{x-a}$ exists

$x \to a+$ means $\forall x \in (a, a+\delta)$

② $f'(b-) = \lim_{x \to b-} \frac{f(x)-f(b)}{x-b}$ exists

Ex: (T or F)

$f(x)$ is differentiable at $a \Leftrightarrow f'(a+), f'(a-)$ exist and $f'(a+) = f'(a-)$

True.

Theorem

$f(x)$ is differentiable at $a \Rightarrow f(x)$ is cont. at a

Proof: $\lim_{x\to a}(f(x)-f(a)) = \lim_{x\to a}\left(\frac{f(x)-f(a)}{x-a}\right)\cdot(x-a)$

Product Theorem $= \lim_{x\to a}\frac{f(x)-f(a)}{x-a}\cdot\lim_{x\to a}(x-a) = f'(a)\cdot 0 = 0$

Linearity Thm $\Rightarrow \lim_{x\to a}f(x) = f(a)$

Theorem (Differentiation Rules)

$f(x)$, $g(x)$ are functions;

a, b are numbers.

① $[af(x)+bg(x)]' = af'(x)+bg'(x)$

② $[f(x)g(x)]' = f'(x)g(x)+f(x)g'(x)$

③ $\left[\frac{f(x)}{g(x)}\right]' = \frac{f'(x)g(x)-f(x)g'(x)}{g(x)^2}$

④ $f(g(x))' = f'(g(x))\,g'(x)$

Theorem

Assume $f(x)$ is differentiable on an open interval I

Then

① $f(x)$ is Locally increasing on I $\Rightarrow f'(x) \geq 0$ on I

② $f(x)$ is Locally decreasing on I $\Rightarrow f'(x) \leq 0$ on I.

Proof of ①: $f(x)$ is differentiable at $a \in I$

$$\Rightarrow f'(a) = f'(a+) = f'(a-)$$

So only need to show $f'(a+) \geq 0$

$f(x)$ is locally increasing at a

$$\Rightarrow \exists \delta > 0,\ f(x) \text{ is } \nearrow \text{ on } (a-\delta, a+\delta)$$

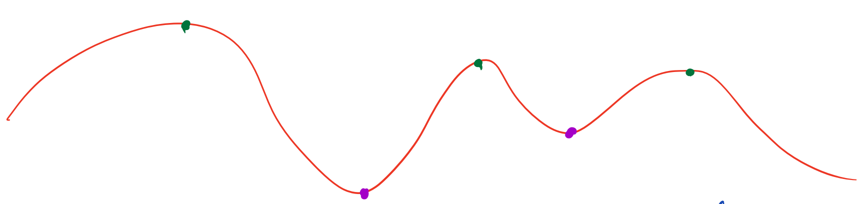
$$\Rightarrow \forall x \in (a, a+\delta),\ f(x) \geq f(a)$$

$$\Rightarrow \frac{f(x)-f(a)}{x-a} \geq 0,\ \forall x \in (a, a+\delta)$$

Limit Location Theorem

$$f'(a+) = \lim_{x \to a+} \frac{f(x)-f(a)}{x-a} \geq 0$$

$$\lim_{x \to a+} \frac{f(x)-f(a)}{x-a} = L : \forall \varepsilon > 0,\ \exists \delta > 0,\ \forall x \in (a, a+\delta)\ \left| \frac{f(x)-f(a)}{x-a} - L \right| < \varepsilon.$$



Def $f(x)$ is defined on an open interval I

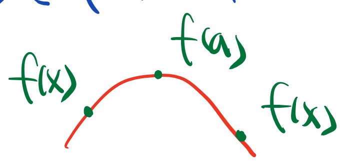
① $a \in I$ is a local max point if

$f(a) \geq f(x)$ for $x \approx a$

$\exists \delta > 0$ s.t. $\forall x \in (a-\delta, a+\delta)$, $f(x) \leq f(a)$.

② $a \in I$ is a local min point if

$f(a) \leq f(x)$ for $x \approx a$



Theorem

Assume $f(x)$ is differentiable on an open interval I

$a \in I$ is a local max (or min) $\Rightarrow f'(a) = 0$

critical point

Proof: Assume a is a local max

$\exists \delta > 0$ s.t. $\forall x \in (a-\delta, a+\delta)$,

$f(x) \leq f(a)$

① $\forall x \in (a-\delta, a)$, $f(x) \leq f(a)$

$$\Rightarrow \frac{f(x)-f(a)}{x-a} \geq 0$$

$$\Rightarrow \lim_{x \to a^-} \frac{f(x)-f(a)}{x-a} \geq 0$$

② $\forall x \in (a, a+\delta)$, $f(x) \leq f(a)$

$$\Rightarrow \frac{f(x)-f(a)}{x-a} \leq 0$$

$$\Rightarrow \lim_{x \to a^+} \frac{f(x)-f(a)}{x-a} \leq 0$$

$$f'(a) = \lim_{x \to a^-} \frac{f(x)-f(a)}{x-a} = \lim_{x \to a^+} \frac{f(x)-f(a)}{x-a}$$

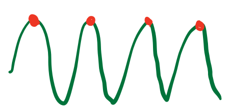
$$\Rightarrow f'(a) = 0.$$

Hw #10

$\boxed{\text{P1}}$ Assume

① $f(x)$ is cont. on compact interval I

② $f(x)$ has infinitely many maximum points



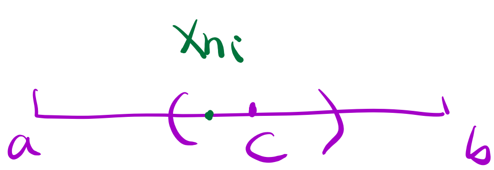
③ $f(x)$ has infinitely many minimum points

④ Between any two max points, there is at least one min point.

Prove that $f(x)$ must be constant.

Idea/Hint: ① Let $\{x_n\} \subset I$ be a sequence of max points

Then I is compact $\Rightarrow$ A subsequence x_{n_i} converges to $c \in I$



$f(x_{n_i}) = M$

$f(x_{n_i}) \to f(c)$

$(c-\delta, c+\delta)$

② Between x_{n_i} and $x_{n_{i+1}}$, there is a min point y_{n_i}

$a_n < b_n < a_{n+1}$

$\downarrow$ a $\quad$ $\downarrow$ a

Then $y_{n_i} \to c$ (Why?)

③ Let $f(x_{n_i}) = M$ (Max value)

$f(y_{n_i}) = m$ (min value)

$$\left.\begin{array}{l} x_{n_i} \to c \Rightarrow f(x_{n_i}) \to f(c) \\ y_{n_i} \to c \Rightarrow f(y_{n_i}) \to f(c) \end{array}\right\} \Rightarrow ?$$

$\boxed{P2}$ (a) Assume

$\{a_n\}$ is Cauchy: $\forall \varepsilon > 0$, $|a_m - a_n| < \varepsilon$, $m, n \gg 1$

① $f(x)$ is uniformly cont. on I

② $\{a_n\}$ is Cauchy

Prove $f(a_n)$ is Cauchy

Idea: $\forall \delta > 0$, $|a_m - a_n| < \delta$, $m, n \gg 1$

$\forall \varepsilon > 0$, $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$ $\quad$ $x = a_m$, $y = a_n$

(b) Use (a) to prove $\frac{1}{x}$ is not uniformly cont. on $(0, 1]$

Idea: find x_n is Cauchy s.t. $\frac{1}{x_n}$ is not Cauchy

$\boxed{P3}$ Use definition and product rule of differentiation to prove quotient rule of differentiation.

$\boxed{P4}$ Assume ① $f(x)$ is differentiable for any x.

② $\forall a, b$, $f(a+b) = f(a) + f(b) + 2ab$

Prove $f'(x) = f'(0) + 2x$

Examples: $f(x) = x^2$

$f(x) = x^2 + x$

Hint: $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{\Delta x \to 0} \frac{f(a+\Delta x) - f(a)}{\Delta x}$

$\boxed{P5}$ $f(x) = \begin{cases} x^2 \sin(\frac{1}{x}) & , x \neq 0 \\ 0 & , x = 0 \end{cases}$

Prove ① $f(x)$ is differentiable for any x

② $f'(x)$ is not continuous

Hint: You can simply state that $x^2 \sin(\frac{1}{x})$ is differentiable at $x \neq 0$ by Chain/Quotient/Product Rule

For ①, only need to consider $\lim_{x \to 0} \frac{f(x)-f(0)}{x-0}$

$$\frac{f(x)-f(0)}{x-0} = x \sin\left(\frac{1}{x}\right) \to 0 \quad (\text{How?})$$

② $x \neq 0$, $f'(x) = 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$

Show $f'(x)$ does not go to $f'(0) = 0$.