Theorem If I is a compact interval,

$f(x)$ is continuous on $I \Rightarrow f(x)$ is uniformly cont. on I.

Proof: Assume $f(x)$ is NOT uniformly continuous

($f(x)$ is uniformly continuous on an interval I if

$\forall \varepsilon > 0, \exists \delta > 0$ s.t.

$\forall x, y \in I$ satisfying $|x-y| \le \delta$, $|f(x)-f(y)| < \varepsilon$.)

" $f(x)$ is not uniformly continuous on I " means

$\exists \varepsilon > 0$, s.t. $\forall \delta > 0$,

$\exists x, y \in I$ satisfying $|x-y| \le \delta$, $|f(x)-f(y)| \ge \varepsilon$

So $\exists \varepsilon > 0$, s.t. for $\delta = \frac{1}{n}$

$\exists x_n, y_n \in I$ s.t. $\begin{cases} |x_n - y_n| \le \delta = \frac{1}{n} \\ |f(x_n) - f(y_n)| \ge \varepsilon. \end{cases}$

I is compact $\Rightarrow \begin{cases} \text{A convergent subsequene } x_{n_i} \\ x_{n_i} \to c \in I \end{cases}$

$|x_{n_i} - y_{n_i}| \le \frac{1}{n_i} \Rightarrow y_{n_i} \to c$

because

$$-\frac{1}{n_i} \le x_{n_i} - y_{n_i} \le \frac{1}{n_i}$$

$$\underset{\displaystyle c}{\underset{\downarrow}{x_{n_i} - \frac{1}{n_i}}} \le y_{n_i} \le \underset{\displaystyle c}{\underset{\downarrow}{x_{n_i} + \frac{1}{n_i}}}$$

Squeeze Thm

$$\left.\begin{array}{l} x_{n_i} \to c \in I \\ f(x) \text{ is continuous on } I \end{array}\right\} \Rightarrow f(x_{n_i}) \to f(c)$$

$$\left.\begin{array}{l} y_{n_i} \to c \in I \\ f(x) \text{ is continuous on } I \end{array}\right\} \Rightarrow f(y_{n_i}) \to f(c)$$

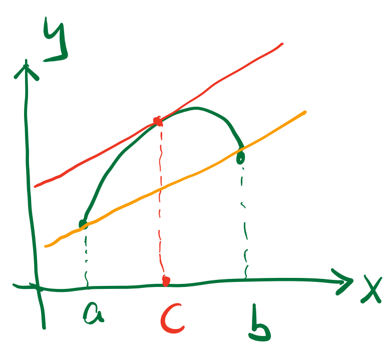
Linearity Thm $\Rightarrow f(x_{n_i}) - f(y_{n_i}) \to 0$

$\Rightarrow |f(x_{n_i}) - f(y_{n_i})| < \varepsilon$ for $i \gg 1$

Contradiction with $|f(x_n) - f(y_n)| \ge \varepsilon, \forall n.$

Chapter 15

Mean Value Theorem

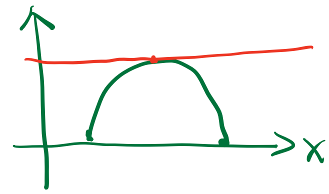


$f(x)$ is continuous on $[a, b]$

and differentiable on (a, b),

then $\exists\, c \in (a, b)$ s.t. $f'(c) = \dfrac{f(b) - f(a)}{b - a}$

Rolle's Theorem



If $f(x)$ is continuous on $[a,b]$ and differentiable on (a,b),

$$f(a)=f(b)=0 \Rightarrow \exists c\in(a,b) \text{ s.t. } f'(c)=0$$

Proof: $f(x)$ is continuous on a compact interval $[a,b]$

$\Rightarrow f(x)$ achieves max/min on $[a,b]$

Let c and d be max point and min point, respectively.

① If $c\in(a,b)$, then $f'(c)=0$ (critical point)

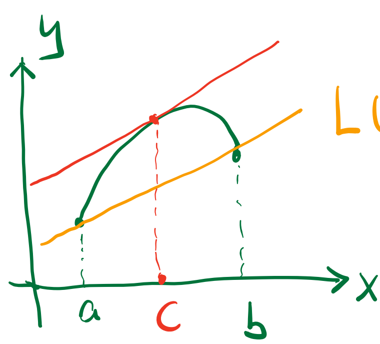
② If $d\in(a,b)$, then $f'(d)=0$

③ If c & d are end points of $[a,b]$, then

$f(x)$ is contant 0 since $\left\{\begin{array}{l} f(a)=f(b)=0 \\ f(c) \text{ is max} \\ f(d) \text{ is min}\end{array}\right.$

thus $\forall c\in(a,b)$, $f'(c)=0$.

Proof of Mean Value Thm:



$$L(x)=f(a)+\frac{f(b)-f(a)}{b-a}(x-a)$$

$$g(x) = f(x) - L(x) = f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x-a)$$

Then $g(a) = g(b) = 0$

$\Rightarrow \exists c \in (a,b)$, $g'(c) = 0$

$$g'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$$

$$\Rightarrow f'(c) = \frac{f(b)-f(a)}{b-a}$$

Theorem Let $f(x)$ be differentiable on interval I.

Then on I,

① $f'(x) > 0, \forall x \in I \Rightarrow f(x)$ is strictly increasing

② $f'(x) \geq 0, \forall x \in I \Rightarrow f(x)$ is increasing

③ $f'(x) = 0, \forall x \in I \Rightarrow f(x)$ is constant.

Proof of ①:

Want to show $\forall a, b \in I$, $a < b \Rightarrow f(a) < f(b)$

Mean Value Thm $\Rightarrow \exists c \in (a,b)$ s.t.

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$f'(c) > 0 \Rightarrow f(b) > f(a)$$

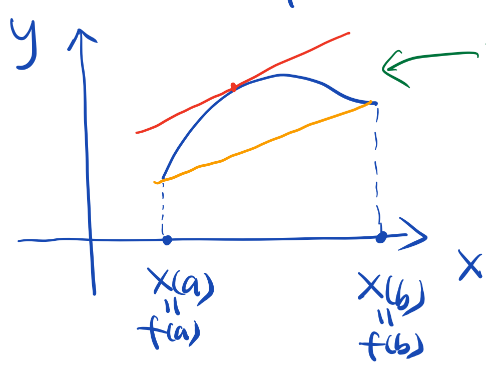
Cauchy Mean-Value Theorem

Assume $f(x)$ and $g(x)$ have continuous derivatives on $[a,b]$, and $g'(x) \neq 0$ on (a,b).

Then $\exists\, c \in (a,b)$ s.t. $\dfrac{f(b)-f(a)}{g(b)-g(a)} = \dfrac{f'(c)}{g'(c)}$.

Proof: $\begin{cases} x = g(t) \\ y = f(t) \end{cases}$

As t varies from a to b, we get a curve.



This function is $y(x)$

Let's called it $h(x)$

Mean Value Thm on $h(x)$

$\Rightarrow h'(c) = \dfrac{h(x(b)) - h(x(a))}{x(b) - x(a)}$

Chain Rule $\Rightarrow \dfrac{d}{dt} y(x(t)) = \dfrac{dy}{dx}\dfrac{dx}{dt}$

$$\begin{cases} x = g(t) \Rightarrow \dfrac{dx}{dt} = g'(t) \\ y = f(t) \Rightarrow \dfrac{dy}{dt} = f'(t) \\ h(x) = y(x) \Rightarrow \dfrac{dy}{dx} = h'(x) \end{cases}$$

$$\Rightarrow f'(t) = h'(x)\, g'(t) \Rightarrow h'(x) = \frac{f'(t)}{g'(t)}$$

$$\left.\begin{matrix}\text{Mean Value Thm} \\ \text{on } h(x)\end{matrix}\right\} \Rightarrow h'(c) = \frac{h[x(b)] - h[x(a)]}{x(b) - x(a)} = \frac{y(b) - y(a)}{x(b) - x(a)}$$

$$\Rightarrow \frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{f(b)-f(a)}$$

<u>Theorem</u> (L'Hospital's Rule for 0/0)

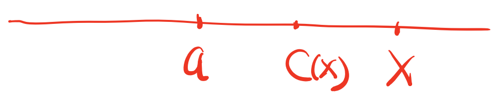
Assume $f(x)$ and $g(x)$ have continuous derivatives and $f(a) = g(a) = 0$, but $\left.\begin{matrix} g'(x) \neq 0 \\ g(x) \neq 0 \end{matrix}\right\}$ for $x \underset{\neq}{\approx} a$.

Then, if the limit on the right exists,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Proof:

For $x \approx a$,



Cauchy Mean Value Thm

$\Rightarrow$ There is c between x and a s.t.

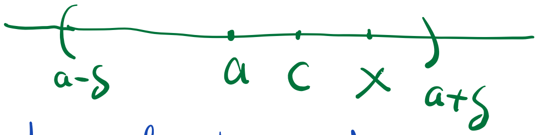
$$\frac{f'(c)}{g'(c)} = \frac{f(x)-f(a)}{g(x)-g(a)} = \frac{f(x)}{g(x)}$$

$$\Rightarrow \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(c(x))}{g'(c(x))} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

because c is between x and a.

$h(x) = \frac{f'(x)}{g'(x)}$ is continuous, and $\lim_{x \to a} h(x)$ exists

Want to show $\lim_{x\to a} h(x) = \lim_{x \to a} h(c(x))$ $\quad c(x)$



Proof: Let $\lim_{x \to a} h(x) = L$

$\forall \varepsilon > 0$, $\exists \delta > 0$, $\forall x \in (a-\delta, a+\delta)$,

$$|h(x) - L| < \varepsilon$$

$\forall x \in (a-\delta, a+\delta)$, $c(x) \in (a-\delta, a+\delta)$

$\Rightarrow |h(c) - L| < \varepsilon$

$\Rightarrow \forall \varepsilon > 0$, $\exists \delta > 0$, s.t. $\forall x \in (a-\delta, a+\delta)$

$$|h(c(x)) - L| < \varepsilon$$

$\Rightarrow \lim_{x \to a} h(c(x)) = L$

L'Hospital's Rule for ∞/∞

If $f(x)$ & $g(x)$ are differentiable, and

$$\begin{cases} f(x) \to +\infty & \text{as } x \to +\infty \\ g(x) \to +\infty & \text{as } x \to +\infty, \end{cases}$$

then, if the limit on the right exists,

$$\lim_{x \to +\infty} \frac{f(x)}{g(x)} = \lim_{x \to +\infty} \frac{f'(x)}{g'(x)}$$

Proof is a practice.

Idea/Hint: Let $\displaystyle\lim_{x \to +\infty} \frac{f'(x)}{g'(x)} = L$

① $\forall \varepsilon > 0$, $\exists a$, s.t. $\forall x > a$, $\dfrac{f'(x)}{g'(x)} \underset{\varepsilon}{\approx} L$.

② Try to show

$$\frac{f(x)}{g(x)} \underset{\varepsilon}{\approx} \frac{f(x) - f(a)}{g(x) - g(a)} \underset{\varepsilon}{\approx} L, \quad \text{for } x \gg 1.$$

③ To show $\dfrac{f(x)}{g(x)} \underset{\varepsilon}{\approx} \dfrac{f(x) - f(a)}{g(x) - g(a)}$

$$f(x) - f(a) = f(x)\left[1 - \frac{f(a)}{f(x)}\right]$$

$$g(x) - f(a) = g(x)\left[1 - \frac{g(a)}{g(x)}\right]$$

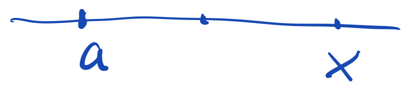
$$\lim_{x \to +\infty} f(x) = +\infty \Rightarrow 1 - \frac{f(a)}{f(x)} \to 1 \quad \text{as } x \to +\infty$$

$$\lim_{x \to +\infty} g(x) = +\infty \Rightarrow 1 - \frac{g(a)}{g(x)} \to 1 \quad \text{as } x \to +\infty$$

$$\Rightarrow \lim_{x\to+\infty} \frac{1-\frac{f(a)}{f(x)}}{1-\frac{g(a)}{g(x)}} = 1$$

$$\frac{f(x)-f(a)}{g(x)-g(a)} = \frac{f(x)}{g(x)} \cdot \frac{1-f(a)/f(x)}{1-g(a)/g(x)} \Rightarrow \frac{f(x)/g(x)}{\frac{f(x)-f(a)}{g(x)-g(a)}} \to 1$$

④ Cauchy Mean Value Thm



$$\left.\begin{aligned}&\Rightarrow \frac{f(x)-f(a)}{g(x)-g(a)} = \frac{f'(c)}{g'(c)} \\ &\frac{f'(x)}{g'(x)} \to L, \quad x\to+\infty\end{aligned}\right\} \Rightarrow \frac{f(x)-f(a)}{g(x)-g(a)} \underset{\varepsilon}{\approx} L.$$

$$\Rightarrow \frac{f(x)-f(a)}{g(x)-g(a)} \Big/ \frac{f'(c)}{g'(c)} = 1$$