Ex: Find an estimate of Geometric Sum/Series

$$\sum_{k=1}^{n} \frac{1}{2^k} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n}$$

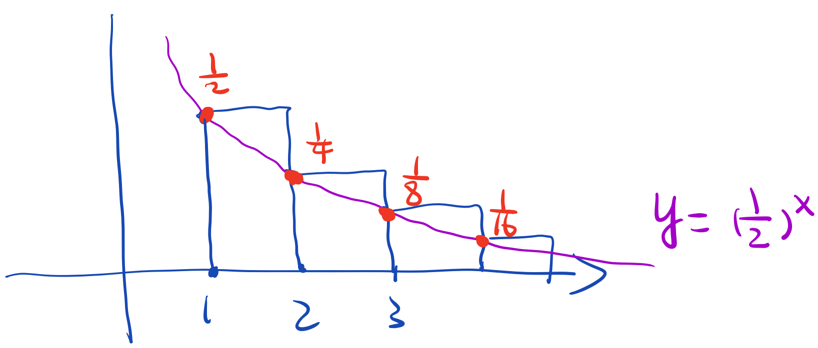
① $$1 + x + x^2 + \cdots + x^n = \frac{1-x^{n+1}}{1-x}$$

$$\Rightarrow \sum_{k=1}^{n} \frac{1}{2^k} = 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} - 1$$

$$= \frac{1-\left(\frac{1}{2}\right)^{n+1}}{1-\frac{1}{2}} - 1$$

$$< \frac{1}{1-\frac{1}{2}} - 1 = 1$$

②



By the picture, we get

$$\sum_{k=1}^{n} \left(\frac{1}{2}\right)^k > \int_1^{n+1} \left(\frac{1}{2}\right)^x dx = \left.\frac{\left(\frac{1}{2}\right)^x}{\ln\left(\frac{1}{2}\right)}\right|_1^{n+1}$$

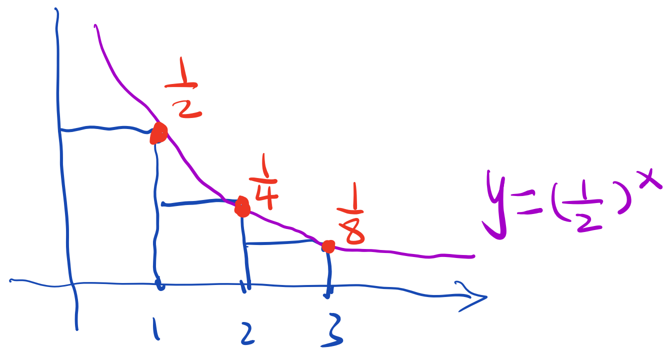
$$\int a^x dx = \frac{a^x}{\ln(a)} + C$$

$$= \frac{\left(\frac{1}{2}\right)^{n+1}}{-\ln 2} - \frac{\frac{1}{2}}{-\ln 2}$$

$$= -\frac{1}{2^{n+1}\ln 2} + \frac{1}{2\ln 2}$$

$$\geq -\frac{1}{4\ln 2} + \frac{1}{2\ln 2} = \frac{1}{4\ln 2}$$

③



$$\sum_{k=1}^{n} \left(\tfrac{1}{2}\right)^k < \int_0^n \left(\tfrac{1}{2}\right)^x dx = \frac{\left(\frac{1}{2}\right)^x}{\ln\left(\frac{1}{2}\right)} \Bigg|_0^n$$

$$= \frac{\left(\frac{1}{2}\right)^n}{-\ln 2} - \frac{\left(\frac{1}{2}\right)^0}{-\ln 2}$$

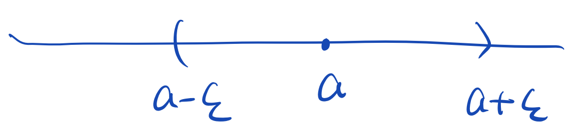
$$= \frac{1}{-2^n \ln 2} + \frac{1}{\ln 2} < \frac{1}{\ln 2}$$

Chapter 2

- Approximation

$$a \underset{\varepsilon}{\approx} b \quad \text{means} \quad |a-b| < \varepsilon$$

$\underset{\varepsilon}{\approx}$



Ex: $\pi = 3.1415926\ldots$

$\pi \underset{0.1}{\approx} 3.1$, $\pi \underset{0.01}{\approx} 3.14$

$\pi \underset{0.05}{\approx} 3.1$, $\pi \underset{0.02}{\approx} 3.14$

• Terminology "for n large"

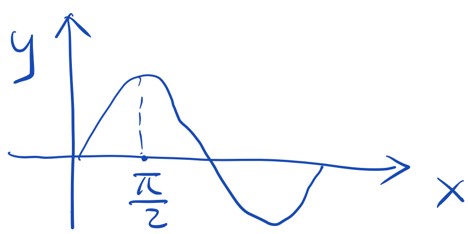
symbol $n>>1$

<u>Def</u> We say $\{a_n\}$ has some property "for n large" if there is some N s.t. a_n has that property for any $n \geq N$.

Example : $a_n = \frac{1}{n}$

$a_n < 0.001, \ n>>1.$

Example: Show $\{\sin\frac{10}{n}\}$ is ↘ for n large

Sol: 

$y = \sin(x)$ is ↗ if $0 < x < \frac{\pi}{2}$

$\Rightarrow \sin\left(\frac{10}{n}\right)$ is ↘ if $\frac{10}{n} < \frac{\pi}{2}$

$\Rightarrow \sin\left(\frac{10}{n}\right)$ is ↘ if $n > \frac{20}{\pi}$

$\Rightarrow \left\{\sin\left(\frac{10}{n}\right)\right\}$ is ↘ , $n>>1$.

Example: If $\{a_n\}$ is bounded above for $n \gg 1$, then $\{a_n\}$ is bounded above.

Proof: $\{a_n\}$ is bounded above for $n \gg 1$

$$\Rightarrow \underbrace{\exists}_{\text{There exists}} N \in \underbrace{\mathbb{N}}_{\text{set of natural numbers}} \text{ and } B \underbrace{\in}_{\text{belongs to}} \mathbb{R} \text{ s.t.}$$

$$a_n \leq B, \quad \forall n \geq N$$

Let $A = \max\{a_0, a_1, \dots, a_N\}$,

then $\quad a_n \leq \max\{A, B\}, \quad \forall n \geq 0$

$\Rightarrow \{a_n\}$ is bounded above.

<u>Theorem</u> (Completeness Theorem)

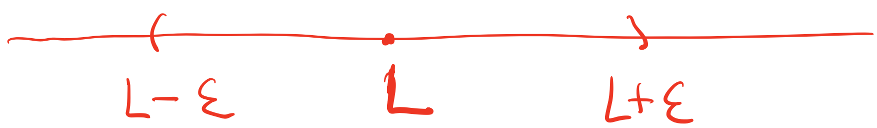
If $\{a_n\}$ is monotone and bounded for $n \gg 1$, then $\{a_n\}$ has a limit.

Chapter 3

<u>Definition</u> L is the limit of $\{a_n\}$ if

$$\forall\ \varepsilon > 0,\quad a_n \underset{\varepsilon}{\approx} L \text{ for } n \gg 1.$$

For any $\varepsilon > 0$, there exists an integer N (depending on ε) s.t. $|a_n - L| < \varepsilon$ for all $n \geq N$.



Intuitively, smaller ε corresponds to larger N

If such L exists, we say $\{a_n\}$ converges;
if not, we say $\{a_n\}$ diverges.

① $\lim\limits_{n\to\infty} a_n = L$ or $\lim a_n = L$

② $a_n \to L$ as $n \to \infty$

③ $a_n \to L$

Example: Show $\lim_{n\to\infty} \frac{n-1}{n+1} = 1$ by definition.

Sol: For any given fixed $\varepsilon > 0$, want to find an N s.t.

$$\left| \frac{n-1}{n+1} - 1 \right| < \varepsilon, \quad \forall n \geq N.$$

$$\left| \frac{n-1}{n+1} - 1 \right| = \left| \frac{n-1-(n+1)}{n+1} \right| = \frac{2}{n+1} < \varepsilon$$

$$\iff n+1 > \frac{2}{\varepsilon} \iff n > \frac{2}{\varepsilon} - 1$$

So $\forall \varepsilon > 0$, let N be a positive integer greater than $\frac{2}{\varepsilon} - 1$,

then for any $n \geq N$,

$$\left| \frac{n-1}{n+1} - 1 \right| = \frac{2}{n+1} \leq \frac{2}{N+1} \leq \varepsilon.$$

By definition, $\frac{n-1}{n+1} \to 1$.

Example: Show $\lim_{n\to\infty} (\sqrt{n+1} - \sqrt{n}) = 0$

Sol: $$A - B = \frac{(A+B)(A-B)}{A+B} = \frac{A^2 - B^2}{A+B}$$

$$\Rightarrow \sqrt{n+1} - \sqrt{n} = \frac{(n+1)-n}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\sqrt{n+1}+\sqrt{n}}$$

For any given $\varepsilon > 0$, want to show

$$\left| \frac{1}{\sqrt{n+1}+\sqrt{n}} - 0 \right| < \varepsilon \quad \text{for } n \gg 1.$$

$$\left| \frac{1}{\sqrt{n+1}+\sqrt{n}} - 0 \right| = \frac{1}{\sqrt{n+1}+\sqrt{n}} < \underbrace{\frac{1}{2\sqrt{n}} < \varepsilon}$$

$$\Updownarrow$$

$$\sqrt{n} > \frac{1}{2\varepsilon}$$

$$\Updownarrow$$

$$n > \frac{1}{4\varepsilon^2}$$

So $\forall \varepsilon > 0$, $\left| \sqrt{n+1} - \sqrt{n} - 0 \right| = \left| \frac{1}{\sqrt{n+1}+\sqrt{n}} - 0 \right|$

$$< \frac{1}{2\sqrt{n}} < \varepsilon \text{ if } n > \frac{1}{4\varepsilon^2}. \quad \#$$

Theorem (Uniqueness of limits)

$\{a_n\}$ has at most one limit:

$$\left.\begin{array}{l} a_n \to L \\ a_n \to L' \end{array}\right\} \Rightarrow L = L'$$

Proof: $\forall \varepsilon > 0$, we have

$$a_n \underset{\varepsilon}{\approx} L, \quad n \gg 1$$

$$a_n \underset{\varepsilon}{\approx} L', \quad n \gg 1$$

$$\Rightarrow \begin{cases} L-\varepsilon < a_n < L+\varepsilon, & n \geq N_1 \\ L'-\varepsilon < a_n < L'+\varepsilon, & n \geq N_2 \end{cases}$$

$$\Rightarrow \begin{cases} L'-\varepsilon < a_n < L+\varepsilon, & n \geq \max\{N_1, N_2\} \\ L-\varepsilon < a_n < L'+\varepsilon, & n \geq \max\{N_1, N_2\} \end{cases}$$

$$\Rightarrow \begin{cases} L' < L+2\varepsilon \\ L-2\varepsilon < L' \end{cases}$$

$$\Rightarrow \forall \varepsilon > 0, \quad L-2\varepsilon < L' < L+2\varepsilon$$

$$L' \underset{2\varepsilon}{\approx} L$$

$\Rightarrow L' = L$.

Theorem $\left.\begin{array}{l}\{a_n\} \text{ is } \nearrow \\ a_n \to L\end{array}\right\} \Rightarrow a_n \le L, \ \forall n.$

Proof:

Assume "$a_n \le L, \forall n$" is not true

$\Rightarrow$ There is one N s.t. $\left.\begin{array}{l} a_N > L \\ a_n \nearrow \end{array}\right\}$

$\Rightarrow \quad a_n \geq a_N > L, \ \forall n \geq N$

$a_n \to L \Rightarrow \forall \varepsilon > 0, \ |a_n - L| < \varepsilon, \ n \gg 1.$

$\Rightarrow \forall \varepsilon > 0, \ a_n < L + \varepsilon, \ n \gg 1$

For $\varepsilon = \frac{a_N - L}{2}$,

$a_n < L + \frac{a_N - L}{2} = \frac{a_N + L}{2}, \ n \gg 1$

But $a_N > L \Rightarrow \frac{a_N + L}{2} < a_N \Rightarrow a_n < a_N, \ n \gg 1$

$a_n \nearrow \Rightarrow a_n \geq a_N, \ n \gg 1.$

$\Rightarrow$ Contradiction.