Definition $\lim_{n\to\infty} a_n = L$ means

$$\forall \varepsilon > 0,\ |a_n - L| < \varepsilon,\ n \gg 1.$$

K-ε principle For proving $\lim_{n\to\infty} a_n = L$, it suffices to show $\forall \varepsilon > 0$, $|a_n - L| < K\varepsilon$, $n \gg 1$

where K is a fixed number.

Infinite Limits

Def $a_n \to +\infty$ means

$$\forall M \geq 0,\ a_n > M,\ n \gg 1.$$

Example: Show $\log(n) \to +\infty$.

Sol: Want to show $\log(n) > M \Leftrightarrow n > e^M$

So $\forall M \geq 0$, $\log(n) > M$ for any $n > e^M$.

Theorem

$$\lim_{n\to\infty} a^n = \begin{cases} +\infty, & \text{if } a > 1 \\ 1, & \text{if } a = 1 \\ 0, & \text{if } |a| < 1. \end{cases}$$

Proof: ① If $a > 1$, let $a = 1 + k$, $k > 0$.

$\forall M \geq 0$, $a^n = (1+k)^n$

Binomial Theorem

$$= 1 + nk + \frac{n(n-1)}{2!}k^2 + \frac{n(n-1)(n-2)}{3!}k^3 + \dots + k^n$$

$$> 1+nk > M \quad , \forall n > \frac{M}{k}$$

② $a=1$ is obvious

③ If $|a|<1$, Let $\frac{1}{|a|} = 1+k$, $k>0$

$$\forall \varepsilon>0, \left| a^n - 0 \right| = |a^n| = |a|^n = \frac{1}{\left(\frac{1}{|a|}\right)^n}$$

$$= \frac{1}{(1+k)^n} = \frac{1}{1+nk+\frac{n(n-1)}{2}k^2+\cdots+k^n}$$

$$< \frac{1}{nk} < \varepsilon \quad , \forall n > \frac{1}{k\varepsilon} .$$

Example: $a_n = \int_0^1 (x^2+2)^n dx$

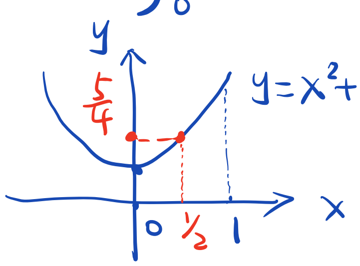
Show $a_n \to +\infty$

Sol: $\forall M \geq 0$,

$$a_n = \int_0^1 (x^2+2)^n dx > \int_0^1 2^n dx = 2^n > M, \forall n > \log_2 M .$$

Example: $a_n = \int_0^1 (x^2+1)^n dx \to +\infty$

Sol:



At $x=\frac{1}{2}$

$y = x^2+1 = \frac{5}{4}$.

$$\int_0^1 (x^2+1)^n dx > \int_{\frac{1}{2}}^1 (x^2+1)^n dx > \int_{\frac{1}{2}}^1 \left(\frac{5}{4}\right)^n dx$$

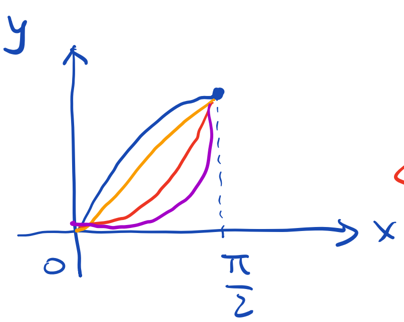
$$= \frac{1}{2}\cdot\left(\frac{5}{4}\right)^n$$

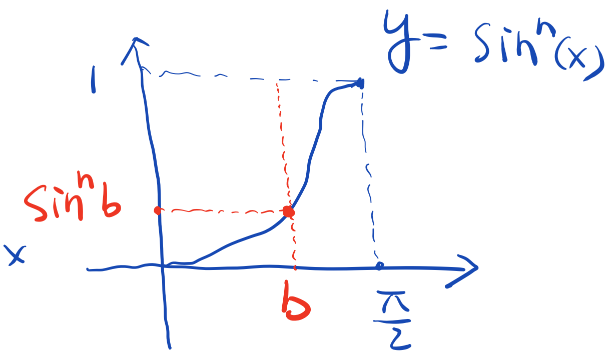
$$\left(\frac{5}{4}\right)^n \to +\infty \Rightarrow \forall M \geq 0,\ \left(\frac{5}{4}\right)^n > M,\ n \gg 1$$

$$\Rightarrow \forall M \geq 0,\ a_n > \frac{1}{2}\left(\frac{5}{4}\right)^n > \frac{M}{2},\ n \gg 1$$

$$\Rightarrow \quad a_n \longrightarrow +\infty.$$

Example: $a_n = \int_0^{\frac{\pi}{2}} \sin^n x\, dx \to 0$





Sol: want to show $a_n < K\varepsilon$, $n \gg 1$.

$$a_n = \int_0^b \sin^n x\, dx + \int_b^{\frac{\pi}{2}} \sin^n x\, dx$$

$$< b \cdot \sin^n b + \left(\frac{\pi}{2} - b\right)$$

For any given fixed small $\varepsilon > 0$, set $b = \frac{\pi}{2} - \varepsilon$

Then $0 < b < \frac{\pi}{2} \Rightarrow \sin b \in (0, 1)$

$$\Rightarrow \quad \sin^n b \to 0,\ n \to \infty$$

$$\Rightarrow \quad \sin^n b = |\sin^n b - 0| < \varepsilon,\ n \gg 1.$$

$$\Rightarrow a_n < b \cdot \sin^n b + \left(\frac{\pi}{2} - b\right)$$

$$< \underbrace{\frac{\pi}{2} \cdot \varepsilon + \varepsilon}_{\substack{\shortparallel \\ \left(\frac{\pi}{2}+1\right)\varepsilon}} \quad , \quad n \gg 1 .$$