Review

① Def $\lim_{n\to\infty} a_n = L$ means

$\forall \varepsilon > 0, \quad |a_n - L| < \varepsilon, \quad n \gg 1.$

② It suffices to show $\forall \varepsilon > 0, \quad |a_n - L| < K\varepsilon, \quad n \gg 1.$

③ Def $a_n \to +\infty$ means

$\forall M \geq 0, \quad a_n > M, \quad n \gg 1.$

④ It suffices to show $\forall M \geq 0, \quad a_n > KM, \quad n \gg 1.$

$K > 0$

⑤ Binomial Theorem

$$(1+k)^n = 1 + nk + \frac{n(n-1)}{2!}k^2 + \frac{n(n-1)(n-2)}{3!}k^3 + \cdots + k^n, \quad \forall k$$

Example $\boxed{1}$ $a > 1, \quad a^n \to +\infty$

$\forall M \geq 0, \quad a = 1 + k, \quad k > 0$

$a^n = (1+k)^n > nk > M, \quad \forall n > \frac{M}{k}.$

$\boxed{2}$ $|a| < 1, \quad a^n \to 0$

$\frac{1}{|a|} = 1 + k, \quad k > 0$

$$\forall \varepsilon > 0, \quad |a^n - 0| = |a|^n = \frac{1}{\left(\frac{1}{|a|}\right)^n} = \frac{1}{(1+k)^n} < \frac{1}{nk} < \varepsilon,$$

$\forall n > \frac{1}{k\varepsilon}.$

$\boxed{3}$ HW#2 P1

$a > 1, \quad \frac{a^n}{n} \to +\infty$

$a = 1+k$, $k > 0$, $\frac{a^n}{n} = \frac{(1+k)^n}{n} > \frac{?}{n}$

$\boxed{4}$ HW#2 P3 $\quad \Rightarrow a > n h_n$

$\Rightarrow h_n < \frac{a}{n} < \varepsilon$, $n \gg 1$.

3. (20 pts) Page 47, 3.4/5. Prove that $a^{\frac{1}{n}} \longrightarrow 1$ if $a > 0$.
Hint: Following the hint in the book: for the case $a > 1$, we know $a^{\frac{1}{n}} > 1$ thus $a^{\frac{1}{n}} = 1 + h_n$ for some $h_n > 0$; then by Binomial Theorem

$|a^{\frac{1}{n}} - 1| = |h_n| < \varepsilon$

$$a = (1+h_n)^n = 1 + nh_n + \frac{1}{2}n(n-1)h_n^2 + \cdots + h_n^n. > nh_n$$

Try to derive an inequality from the equation above so that you can show $h_n \to 0$ by definition.

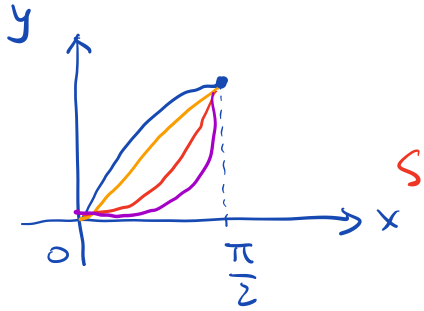
$\boxed{5}$ HW#2 P4

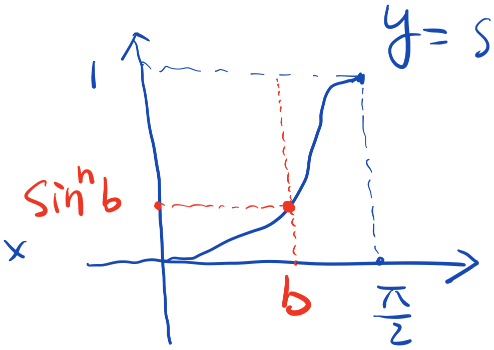
4. (20 pts) Page 59, Problem 4-1. Prove that $n^{\frac{1}{n}} \to 1$.
Hint: Let $e_n = n^{\frac{1}{n}} - 1$, then by $n^{\frac{1}{n}} = e_n + 1$ and Binomial Theorem, we get

$$n = (1+e_n)^n = 1 + ne_n + \frac{1}{2}n(n-1)e_n^2 + \cdots + e_n^n. > \boxed{?}$$

We know $e_n > 0$. Try to derive an inequality from the equation above so that you can show $e_n \to 0$ by definition.

⑥ Example: $a_n = \int_0^{\frac{\pi}{2}} \sin^n x \, dx \to 0$





Sol: want to show $a_n < K\varepsilon$, $n \gg 1$.

$$a_n = \int_0^b \sin^n x \, dx + \int_b^{\frac{\pi}{2}} \sin^n x \, dx$$

$$(\text{by pic above}) < b\cdot \sin^n b + \left(\frac{\pi}{2} - b\right)$$

For any given fixed small $\varepsilon > 0$, set $b = \frac{\pi}{2} - \varepsilon$

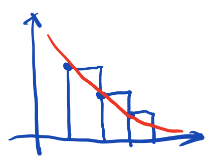
Then $0 < b < \frac{\pi}{2} \Rightarrow \sin b \in (0, 1)$

$$\Rightarrow \sin^n b \to 0, \quad n \to \infty$$

$$\Rightarrow \sin^n b = |\sin^n b - 0| < \varepsilon, \quad n \gg 1.$$

$$\Rightarrow a_n < b\cdot \sin^n b + \left(\frac{\pi}{2} - b\right)$$

$$< \underbrace{\frac{\pi}{2}\cdot \varepsilon + \varepsilon}_{= \left(\frac{\pi}{2}+1\right)\varepsilon}, \quad n \gg 1.$$



Example: HW#2 P5 $\quad a_n = \frac{1}{n}$

$$b_n = \frac{1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}}{n}$$

5. (20 pts) Page 48, Problem 3-1.
For a given sequence $\{a_n\}$, another sequence $\{b_n\}$ is defined as its average:

$$b_n = \frac{a_1 + a_2 + \cdots + a_n}{n}$$

$< \varepsilon \Leftrightarrow n > \frac{a_1 + a_2 + \cdots a_{N-1}}{\varepsilon}$

(a) Prove that if $a_n \to 0$, then $b_n \to 0$. $\quad \frac{a_1 + a_2 + \cdots + a_{N-1}}{n} + \frac{a_N + \cdots + a_n}{n}$

N_ε

Hint: $a_n \to 0$ means that for any fixed $\epsilon > 0$, there is an N s.t. $|a_n| < \epsilon$ for any $n \geq N$. Thus $\left|\frac{a_{N+1} + a_{N+2} + \cdots + a_n}{n}\right|$ is smaller than ϵ. Show that if n is large enough (find that index) then the other part of b_n is also smaller than ϵ (this is possible because N is fixed for fixed ϵ).

Section 4.1

Theorem (Error-form Principle) If $a_n = L + e_n$, then

$$a_n \to L \Longleftrightarrow e_n \to 0$$

Proof: $\quad a_n \to L$ means $\forall \varepsilon > 0, \ |a_n - L| < \varepsilon, \ n \gg 1$

$\quad e_n \to 0$ means $\forall \varepsilon > 0, \ |e_n - 0| < \varepsilon, \ n \gg 1$

Example : $a_n = 1 + a + \cdots + a^n$

Prove $a_n \to \frac{1}{1-a}$, if $|a|<1$

Sol: $a_n = \frac{1-a^{n+1}}{1-a} = \frac{1}{1-a} - \frac{a^{n+1}}{1-a}$

$|a|<1 \Rightarrow a^n \to 0 \Rightarrow \forall \varepsilon,\ |a^n - 0| < \varepsilon,\ n \gg 1$

$\Rightarrow \forall \varepsilon,\ \left|-\frac{a^{n+1}}{1-a} - 0\right| = \frac{|a|}{|1-a|} \cdot |a^n| < \frac{|a|}{|1-a|} \varepsilon,\ n \gg 1$

$\Rightarrow -\frac{a^{n+1}}{1-a} \to 0$

$\Rightarrow a_n \to \frac{1}{1-a}$.

Example: $b_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^{n-1}}{n}$

Show $b_n \to \ln 2$ $\quad 1 + a + a^2 + \cdots + a^n = \frac{1-a^{n+1}}{1-a}$

Replace a by $(-u)$ in previous ex, we get

$$1 - u + u^2 - \cdots + (-1)^{n-1} u^{n-1} = \frac{1}{1+u} - (-1)^n \frac{u^n}{1+u},$$

$u \neq -1$

Integrate both sides from 0 to 1

$$\int_0^1 \left[1 - u + u^2 - \cdots + (-1)^{n-1} u^{n-1}\right] du = \int_0^1 \left(\frac{1}{1+u} - (-1)^n \frac{u^n}{1+u}\right) du$$

$$\Rightarrow \left[u - \frac{u^2}{2} + \frac{u^3}{3} + \cdots + (-1)^{n-1} \frac{u^n}{n}\right]\Bigg|_0^1 = \ln(u+1)\Big|_0^1 \pm \int_0^1 \frac{u^n}{1+u} du$$

$$\Rightarrow 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^{n-1}}{n} = \ln 2 \pm \int_0^1 \frac{u^n}{1+u} du$$

Only need to show $e_n = \pm \int_0^1 \frac{u^n}{1+u} du \to 0$

$$\frac{u^n}{1+u} \leq u^n, \quad 0 \leq u \leq 1$$

$$\Rightarrow |e_n| = \int_0^1 \frac{u^n}{1+u} du = \int_0^1 u^n du = \frac{1}{n+1} < \varepsilon,$$

$$\forall n > \frac{1}{\varepsilon}.$$

Chapter 5

$\boxed{\text{Theorem}}$ Assume $a_n \to L$

$b_n \to M$.

Then

① $r a_n + s b_n \to rL + sM$

② $a_n b_n \to LM$

③ $\frac{b_n}{a_n} \to \frac{M}{L}$, if $L, a_n \neq 0, \forall n$.

Proof of ②: $a_n = L + e_n$, $e_n \to 0$

$b_n = M + e_n'$, $e_n' \to 0$

$$a_n b_n = (L + e_n)(M + e_n')$$

$$= LM + e_n M + e_n' L + e_n e_n'$$

For any $\underset{\text{small}}{\varepsilon} > 0$, $\begin{cases} e_n \to 0 \Rightarrow |e_n| < \varepsilon, \forall n \geq N_1. \\ e_n' \to 0 \Rightarrow |e_n'| < \varepsilon, \forall n \geq N_2. \end{cases}$

Let $N = \max\{N_1, N_2\}$, then $\forall n \geq N$,

$$|a_n b_n - LM| = |e_n M + e_n' L + e_n e_n'|$$
$$\leq |e_n M| + |e_n' L| + |e_n e_n'|$$
$$< M\varepsilon + L\varepsilon + \varepsilon \cdot \varepsilon$$
$$< (M + L + 1)\varepsilon$$

for $\varepsilon < 1$.