Chapter 5

$\boxed{\text{Theorem 1}}$ Assume $a_n \to L$
$b_n \to M$.

Then

① $r a_n + s b_n \to rL + sM$

② $a_n b_n \to LM$

③ $\frac{b_n}{a_n} \to \frac{M}{L}$, if $L, a_n \neq 0, \forall n$.

Example: Show $\lim_{n\to\infty} \frac{3n^2-2n-1}{n^2+1} = 3$

Sol: $\frac{3n^2-2n-1}{n^2+1} = \frac{3-\frac{2}{n}-\frac{1}{n^2}}{1+\frac{1}{n^2}}$

$\forall \varepsilon > 0, \left|\frac{1}{n}\right| < \varepsilon, \forall n > \frac{1}{\varepsilon}$

$\Rightarrow \frac{1}{n} \to 0 \Rightarrow \begin{cases} \frac{1}{n^2} \to 0 \\ -\frac{2}{n} \to 0 \end{cases}$

So $\frac{3n^2-2n-1}{n^2+1} = \frac{3-\frac{2}{n}-\frac{1}{n^2}}{1+\frac{1}{n^2}} \to 3$.

$\boxed{\text{Theorem 2}}$

① $a_n \to +\infty$, $\begin{cases} b_n \to +\infty \text{ or} \\ b_n \to L \text{ or} \\ b_n \text{ bounded below} \end{cases} \Rightarrow a_n + b_n \to +\infty$

② $a_n \to +\infty$, $\begin{cases} b_n \to +\infty \text{ or} \\ b_n \to L > 0 \text{ or} \\ b_n \geq K > 0,\ n \gg 1 \end{cases} \Rightarrow a_n \cdot b_n \to +\infty$

③ $a_n \to +\infty \Rightarrow \frac{1}{a_n} \to 0$

④ $\left.\begin{array}{l} a_n \to 0 \\ a_n > 0,\ n \gg 1 \end{array}\right\} \Rightarrow \frac{1}{a_n} \to +\infty$

Proof of ③: $a_n \to +\infty$

$\Rightarrow \forall M \geq 0,\ a_n > M,\ n \gg 1$

$\Rightarrow$ For any $\varepsilon > 0$, let $M = \frac{1}{\varepsilon}$, then

$a_n > M = \frac{1}{\varepsilon},\ n \gg 1$

$\Rightarrow \forall \varepsilon > 0,\ 0 < \frac{1}{a_n} < \varepsilon,\ n \gg 1.$

$\Rightarrow \frac{1}{a_n} \to 0$

Example: Find $\lim\limits_{n \to \infty} n(a + \cos(n\pi))$

Sol: ① If $a > 1$, then $a + \cos(n\pi) \geq a - 1 > 0$

Theorem 2 ② $\Rightarrow n(a-1) \to +\infty$

$\Rightarrow \forall M > 0,\ n(a-1) > M,\ n \gg 1$

$\Rightarrow \forall M > 0,\ n(a + \cos(n\pi)) > M,\ n \gg 1$

$\Rightarrow n(a + \cos(n\pi)) \to +\infty$

② If $a < -1$, similary, $n(a + \cos(n\pi)) \to -\infty$

$a_n \to -\infty$ means $\forall M \geq 0$, $a_n < -M$, $n \gg 1$.

③ If $a=1$ or $a=-1$, alternating between 0 and $\pm 2n$, so no limits

$n(1+\cos(n\pi))$

$0, 2\cdot 2, 0, 4\cdot 2$

"a_n has no limits" means ?

1) There is no $L \in \mathbb{R}$ s.t. $a_n \to L$

2) There is no $L \in \mathbb{R}$ s.t. $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$, s.t. $\forall n > N$ $|a_n - L| < \varepsilon$,

3) For any $L \in \mathbb{R}$, $a_n \longrightarrow L$ is false

4) $\forall L \in \mathbb{R}$, $\exists \varepsilon > 0$, s.t. $\forall N \in \mathbb{N}$, $\exists n > N$, $|a_n - L| \geq \varepsilon$.

Negation of "$\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $|a_n - L| < \varepsilon$, $\forall n > N$"

is "$\exists \varepsilon > 0$, $\forall N \in \mathbb{N}$, $\exists n > N$ s.t. $|a_n - L| \geq \varepsilon$"

④ If $|a| < 1$, the terms alternate in signs, but tend to $+\infty$ in size, so no limit.

(we can give a rigorous proof later after proving more Theorem)

$\boxed{\text{Squeeze/Sandwich Theorem}}$

① If $a_n \leq b_n \leq c_n$, $n \gg 1$,

then $\left.\begin{array}{l} a_n \to L \\ c_n \to L \end{array}\right\} \Rightarrow b_n \to L$

② $\left.\begin{array}{l} a_n \to +\infty \\ b_n \geq a_n,\ n \gg 1 \end{array}\right\} \Rightarrow b_n \to +\infty$

Proof of ①: $\left.\begin{array}{l} a_n \to L \\ c_n \to L \end{array}\right\} \Rightarrow \forall \varepsilon > 0,\quad \begin{array}{l} |a_n - L| < \varepsilon, \forall n > N_1 \\ |c_n - L| < \varepsilon, \forall n > N_2 \end{array}$

Let $N = \max\{N_1, N_2\}$, then

$$\begin{cases} |a_n - L| < \varepsilon \\ |c_n - L| < \varepsilon \end{cases}, \quad \forall n > N$$

$$\Rightarrow \begin{cases} L-\varepsilon < a_n < L+\varepsilon \\ L-\varepsilon < c_n < L+\varepsilon \end{cases}, \quad \forall n > N$$

$\exists N_3, \quad a_n \le b_n \le c_n, \quad \forall n > N_3$

Let $N_4 = \max\{N, N_3\}$, then

$L-\varepsilon < a_n \le b_n \le c_n < L+\varepsilon, \quad \forall n > N_4$

$\Rightarrow L-\varepsilon < b_n < L+\varepsilon, \quad n >> 1$

$\Rightarrow \quad b_n \to L.$

Example: Show $[2+\cos(na)]^{\frac{1}{n}} \to 1$

where a is a fixed number.

Sol: ① $a^{\frac{1}{n}} > b^{\frac{1}{n}} \xRightarrow{\text{Multiply } n \text{ copies}} a > b$
$a, b > 0$

The contrapositive is $a \le b \Rightarrow a^{\frac{1}{n}} \le b^{\frac{1}{n}}$

② $-1 \le \cos(na) \le 1$

$\Rightarrow \quad 1 \le 2 + \cos(na) \le 3$

$$\Rightarrow \quad 1 \le [2+\cos(na)]^{\frac{1}{n}} \le 3^{\frac{1}{n}}$$

③ HW#2 P3 $\Rightarrow 3^{\frac{1}{n}} \to 1$

④ Squeeze Theorem $\Rightarrow [2+\cos(na)]^{\frac{1}{n}} \to 1$.

Example: Prove $a>1 \Rightarrow a^n \to +\infty$

Sol: $a>1 \Rightarrow a = 1+k, \ k>0$

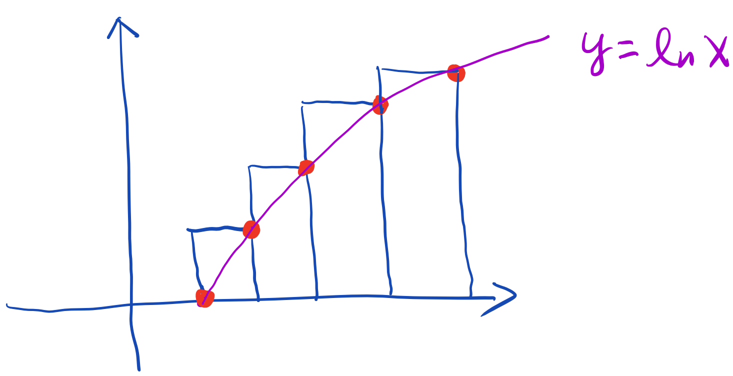
$\Rightarrow a^n > 1+nk$ by Binomial Thm

$1+nk \to +\infty \Rightarrow a^n \to +\infty$ by Squeeze Thm.

Example: Show $\dfrac{\ln n!}{n \ln n} \to 1$

Sol: $\ln n! = \ln(1\cdot 2\cdot 3\cdots n)$

$= \ln 1 + \ln 2 + \cdots + \ln n < n\ln n$



Picture $\Rightarrow \ln 1 + \ln 2 + \cdots + \ln n$

$> \int_1^n \ln x\,dx = (x\ln x - x)\Big|_1^n$

$= n\ln n - n + 1$

$$\Rightarrow \quad 1-\frac{1}{\ln n}+\frac{1}{n\ln n}<\frac{\ln n!}{n\ln n}<\frac{n\ln n}{n\ln n}=1$$

$$\left.\begin{array}{l}\ln n\to+\infty\\ n\to+\infty\end{array}\right\}\Rightarrow 1-\frac{1}{\ln n}+\frac{1}{n\ln n}\to 1$$

Squeeze Thm $\Rightarrow \dfrac{\ln n!}{n\ln n}\to 1.$

Theorem (Location Theorem)

If $a_n\to L$, then

$a_n\leq M,\ n\gg1\Rightarrow L\leq M$

$a_n\geq M,\ n\gg1\Rightarrow L\geq M.$

Proof: $a_n\to L\Rightarrow L-\varepsilon<a_n<L+\varepsilon,\ n\gg1$

$a_n\leq M,\ n\gg1$

$\Rightarrow \forall\varepsilon>0,\quad L-\varepsilon<M$

$\Rightarrow \quad L\leq M$

Proof of the claim by contradiction:

If $L>M$, pick $\varepsilon=L-M>0$,

$L-\varepsilon=M$, contradiction

with $L-\varepsilon<M$.

Theorem If $a_n \to L$,

$$L < M \Rightarrow a_n < M, \quad n \gg 1$$

$$L > M \Rightarrow a_n > M, \quad n \gg 1.$$