$\boxed{\text{Review}}$

Def $\{a_n\}$ converges if there is a $L \in \mathbb{R}$ s.t. $a_n \to L$

$\exists L \in \mathbb{R}$ s.t. $\begin{cases} \forall \varepsilon > 0, \ |a_n - L| < \varepsilon, \ n \gg 1. \\ \forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N}, |a_n - L| < \varepsilon, \forall n > N_\varepsilon \end{cases}$

we say $\lim_{n\to\infty} a_n$ exists if $a_n \to L$ for some $L \in \mathbb{R}$.

Def $\{a_n\}$ diverges if there is no $L \in \mathbb{R}$ s.t. $a_n \to L$

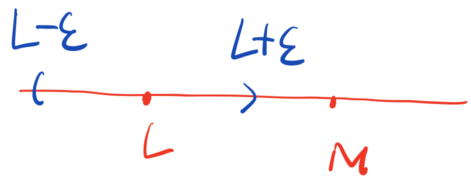
$\forall L \in \mathbb{R}$, $a_n \to L$ is false

$\forall L \in \mathbb{R}, \exists \varepsilon > 0$, s.t. $\forall N \in \mathbb{N}, \exists n > N,$ $|a_n - L| \geq \varepsilon$.

depending on L

Remark: by this definition of convergence, $a_n \to +\infty$ is a divergent sequence, e.g. if $a_n = n^2$, both a_n and $|a_{n+1} - a_n|$ become arbitrarily large, so it is divergent. We denote it as $a_n \to +\infty$ only for convenience.

Review of Theorems



$\boxed{\text{Theorem}}$ If $a_n \to L$,

$$L < M \Rightarrow a_n < M, \ n \gg 1$$

$$L > M \Rightarrow a_n > M, \ n \gg 1.$$

$\boxed{\text{Theorem (Location Theorem)}}$

If $a_n \to L$, then $\begin{cases} a_n \le M, \ n \gg 1 \Rightarrow L \le M \\ a_n \geq M, \ n \gg 1 \Rightarrow L \geq M. \end{cases}$

$\boxed{\text{Theorem}}$ $\{a_n\} \nearrow$, $a_n \to L \Rightarrow a_n \le L$, $\forall n$

$\boxed{\text{Squeeze/Sandwich Theorem}}$

① If $a_n \le b_n \le c_n$, $n \gg 1$,

then $\left.\begin{matrix} a_n \to L \\ c_n \to L \end{matrix}\right\} \Rightarrow b_n \to L$

② $\left.\begin{matrix} a_n \to +\infty \\ b_n \geq a_n, \ n \gg 1 \end{matrix}\right\} \Rightarrow b_n \to +\infty$

$\boxed{\text{Uniqueness Theorem}}$ If $\{a_n\}$ converges, then $L = \lim\limits_{n\to\infty} a_n$ is unique.

$\boxed{\text{Completeness Theorem}}$ If $\{a_n\}$ is bounded and monotone, then $\{a_n\}$ converges.

Contrapositive: $\forall L \in \mathbb{R}$, $\exists \varepsilon > 0$, s.t. $\forall N \in \mathbb{N}$, $\exists n > N$, $|a_n - L| \geq \varepsilon$.

$\Rightarrow$ either $\forall M \geq 0$, $\exists n \in \mathbb{N}$ s.t. $|a_n| > M$

or $\exists n$ s.t. $a_n > a_{n+1}$ and $\exists k$ s.t. $a_k < a_{k+1}$.

<u>Def</u> A subsequence of $\{a_n\}$ is

$$a_{n_1}, a_{n_2}, a_{n_3}, \cdots, a_{n_i}, \cdots$$

where $n_1 < n_2 < \cdots$

Example: $\{\frac{1}{2n}\}$ is a subsequence of $\{\frac{1}{n}\}$

$\boxed{\text{Subsequence Theorem}}$

① $\lim\limits_{n\to\infty} a_n = L \Rightarrow \lim\limits_{i\to\infty} a_{n_i} = L$ for any subsequence $\{a_{n_i}\}$

② $\lim\limits_{n\to\infty} a_n = +\infty \Rightarrow \lim\limits_{i\to\infty} a_{n_i} = +\infty$ for any subsequence $\{a_{n_i}\}$

Proof of ①: $a_n \to L \Rightarrow \forall \varepsilon > 0$, $\exists N_\varepsilon \in \mathbb{N}$ s.t. $|a_n - L| < \varepsilon$, $\forall n > N_\varepsilon$

$\{a_{n_i}\}$ is a subsequence $\Rightarrow n_i \geq i$

$\Rightarrow$ If $i > N$, then $n_i > N$

$\Rightarrow |a_{n_i} - L| < \varepsilon$, $\forall i > N_\varepsilon$ (because $n_i \geq i > N_\varepsilon$)

$\Rightarrow \lim\limits_{i\to\infty} a_{n_i} = L$

Example: Prove that $\lim\limits_{n\to\infty} \sin\frac{n\pi}{2}$ does not exist

Sol: $a_n = \sin\frac{n\pi}{2}$

Two subsequences:

1) $a_{2k} = \sin(k\pi) \equiv 0 \to 0, \quad k \to \infty$

2) $a_{4k+1} = \sin(2k\pi + \frac{\pi}{2}) \equiv 1 \to 1, \quad k \to \infty$

If $a_n \to L$, then $\begin{cases} a_{2k} \to L \\ a_{4k+1} \to L \end{cases}$

Any convergent sequence has a unique limit (uniqueness theorem).

So it is a contradiction.

Example: Prove $\left.\begin{array}{l} a_n \to 0 \\ b_n \text{ is bounded} \end{array}\right\} \Rightarrow a_n b_n \to 0$

A wrong sol: $C \leq b_n \leq B$

$$\cancel{\Rightarrow} \; a_n C \leq a_n b_n \leq a_n B$$

$$a_n \to 0 \Rightarrow \begin{cases} a_n C \to 0 \\ a_n B \to 0 \end{cases}$$

Squeeze Theorem $\Rightarrow a_n b_n \to 0$

Sol: ① $a_n \to 0 \Rightarrow \forall \varepsilon > 0, \; |a_n - 0| < \varepsilon, \; n >> 1$

$$\Rightarrow \forall \varepsilon > 0, \; \big||a_n| - 0\big| = |a_n| < \varepsilon, \; n >> 1$$

$$\Rightarrow \quad |a_n| \to 0$$

② $\quad C \le b_n \le B$

$$\Rightarrow \quad 0 \le |b_n| \le K = \max\{|B|, |C|\}$$

$$\Rightarrow \quad 0 \le |a_n b_n| \le |a_n| \cdot K$$

$$|a_n| \to 0 \Rightarrow |a_n| \cdot K \to 0$$

Squeeze Thm $\Rightarrow |a_n b_n| \to 0$

$$-|a_n b_n| \le a_n b_n \le |a_n b_n|$$

Squeeze Thm $\Rightarrow \quad a_n b_n \to 0$

Example: Prove $\left.\begin{array}{l} a_n \to L \\ a_n, L \neq 0 \end{array}\right\} \Rightarrow \frac{1}{a_n} \to \frac{1}{L}$

Location Theorem If $\{a_n\}$ converges,

$$\lim_{n\to\infty} a_n > M \Rightarrow a_n > M, \ n \gg 1$$

Sol: ① Assume $L > 0$.

$$a_n \to L > \frac{L}{2} \Rightarrow a_n > \frac{L}{2}, \ n > N.$$

For any given fixed $\varepsilon > 0$, $|a_n - L| < \varepsilon$, $n > N_\varepsilon$.

$$\forall n > \max\{N, N_\varepsilon\}, \ \left|\frac{1}{a_n} - \frac{1}{L}\right| = \frac{|L - a_n|}{|a_n L|} = \frac{|a_n - L|}{a_n L} < \frac{\varepsilon}{(\frac{L}{2})L}.$$

$$\Rightarrow \left|\frac{1}{a_n} - \frac{1}{L}\right| < \frac{2}{L^2}\varepsilon, \quad n \gg 1$$

$$\Rightarrow \frac{1}{a_n} \to \frac{1}{L}$$

② Assume $L < 0$

$$a_n \to L \Rightarrow -a_n \to -L \overset{①}{\Rightarrow} -\frac{1}{a_n} \to -\frac{1}{L}$$

$$\Rightarrow \frac{1}{a_n} \to \frac{1}{L}.$$

$\boxed{\text{HW\#3 P3}}$

$$|a_{n+1}| < \frac{1}{2}|a_n|, \quad \forall n \geq N$$

$$\left.\begin{aligned} |a_{\cancel{N+1}}| &< \tfrac{1}{2}|a_N| \\ |a_{\cancel{N+2}}| &< \tfrac{1}{2}|a_{\cancel{N+1}}| \\ |a_{N+3}| &< \tfrac{1}{2}|a_{\cancel{N+2}}| \\ &\vdots \\ |a_{N+k}| &< \tfrac{1}{2}|a_{N+k-1}| \end{aligned}\right\} \Rightarrow \underbrace{|a_{N+k}|}_{\text{subsequence of } \{a_n\}} < \frac{1}{2^k}|a_N|$$

$\boxed{\text{HW\#3 P4}}$

$a_n \geq 0$

$a_n \to L \Rightarrow \sqrt{a_n} \to \sqrt{L}$

$\left.\begin{array}{l}\text{Negation of } \sqrt{a_n} \to \sqrt{L} \\ a_n \geq 0\end{array}\right\} \Rightarrow \text{Negation of } a_n \to L$

" $\sqrt{a_n} \to \sqrt{L}$ " is

$\forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N}$ s.t. $\forall n > N_\varepsilon, |\sqrt{a_n} - \sqrt{L}| < \varepsilon$

"Negation of $\sqrt{a_n} \to \sqrt{L}$" is

$\exists \varepsilon > 0, \forall N \in \mathbb{N}, \exists n_N > N, |\sqrt{a_n} - \sqrt{L}| \geq \varepsilon.$

We can use c instead of ε

$\exists c > 0, \forall N \in \mathbb{N}, \exists n_N > N, |\sqrt{a_n} - \sqrt{L}| \geq c.$

Want to show "Negation of $a_n \to L$":

$\forall N \in \mathbb{N}, \exists n_N > N, |a_n - L| \geq$ some fixed positive number

$$\sqrt{a_n} > \boxed{?} \Rightarrow \left.\begin{array}{l}|\sqrt{a_n} - \sqrt{L}| \geq c \\ |\sqrt{a_n} + \sqrt{L}| \geq \boxed{?}\end{array}\right\} \Rightarrow |a_n - L| \geq \boxed{?}$$

$\Uparrow$

$\sqrt{a_n} \to \sqrt{L}$

Chapter 6

$\boxed{\text{Nested Interval Theorem}}$

If $[a_0, b_0] \supseteq [a_1, b_1] \supseteq [a_2, b_2] \supseteq \cdots$

then $\lim_{n\to\infty}(b_n - a_n) = 0 \Rightarrow \exists L$, $a_n \to L$, $b_n \to L$.

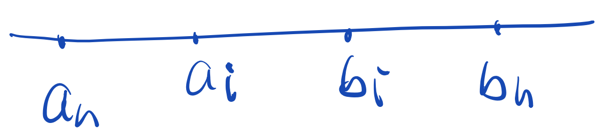
Proof:

<u>Step 1</u> Claim $a_n \le b_i$, $\forall n, \forall i$

because $\begin{cases} i \le n \Rightarrow a_n \le b_n < b_i \\ i > n \Rightarrow a_n \le a_i \le b_i \end{cases}$

$i \le n$:

$i > n$:



<u>Step 2</u> $\{a_n\}$ is $\uparrow$

$a_n \le b_1 \Rightarrow \{a_n\}$ is bounded above

Completeness Thm $\Rightarrow a_n \to L$ for some $L \in \mathbb{R}$.

<u>Step 3</u>

<u>Theorem</u> $\left.\begin{array}{l}\{a_n\} \text{ is } \uparrow \\ a_n \to L\end{array}\right\} \Rightarrow a_n \le L, \forall n.$

$\Rightarrow a_n \le L, \forall n$

$a_n \le b_i \Rightarrow \lim\limits_{n\to\infty} a_n \le b_i$

$\Rightarrow L \le b_i, \forall i$

$\Rightarrow a_n \le L \le b_n, \forall n$

<u>Step 4</u> $\left.\begin{array}{l}\{b_n\} \text{ is } \downarrow \\ b_n \ge a_1\end{array}\right\} \Rightarrow \lim\limits_{n\to\infty} b_n = M$ for some M.

$b_n \ge a_k \Rightarrow \lim\limits_{n\to\infty} b_n \ge a_k \Rightarrow M \ge a_k, \forall k.$

$$\left.\begin{array}{l} b_n \searrow \\ b_n \to M \end{array}\right\} \Rightarrow M \leq b_n, \forall n.$$

<u>Step 5</u> $L, M \in [a_n, b_n], \forall n$

Then want to show $L = M$.

$$L, M \in [a_n, b_n] \Rightarrow |L-M| \leq b_n - a_n$$

$$\text{(Location Thm)} \Rightarrow |L-M| \leq \underbrace{\lim_{n\to\infty}(b_n - a_n)}_{=0}$$

$$\Rightarrow L = M.$$