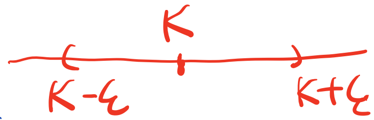
Chapter 6



Def $K \in \mathbb{R}$ is a cluster point of $\{a_n\}$ if $\forall \varepsilon > 0$, $|a_n - K| < \varepsilon$ for infinitely many n.

Example: ① $a_n = (-1)^n$ has two cluster points 1 and -1

② $a_n \to L \Rightarrow L$ is a cluster point of $\{a_n\}$

③ $a_n = n$, $a_n \to +\infty$

a_n has no cluster points

③ is intuitive but how to prove it?

Question: what is the negation of "K is a cluster point of $\{a_n\}$"?

Negation of "$\forall \varepsilon > 0$, $|a_n - K| < \varepsilon$ for infinitely many n"

Wrong negation: 1) $\exists \varepsilon > 0$, $|a_n - K| \geq \varepsilon$ for infinitely many n

counter example is

$K = -1$ for $a_n = (-1)^n$, $|a_{2k} - K| = 2$, $\forall k$

Wrong negation: 2) $\exists \varepsilon>0$, $|a_n-K|\geq\varepsilon$ for finitely many n

counter example is

$K=0$ for $a_n=\frac{1}{n}$, $\varepsilon=1$, $|a_n-K|\geq 1$ only for $n=1$.

$\boxed{\text{To find correct negation, we shoud use "}\forall, \exists\text{" for expressing "infinitely many'}}$

Step 0: meaning of "K is a cluster point of a_n"

1) $\forall \varepsilon>0$, $|a_n-K|<\varepsilon$ for infinitely many n.

2) $\forall \varepsilon>0$, $\forall N\in\mathbb{N}$, $\exists n>N$ s.t. $|a_n-K|<\varepsilon$

So "K is NOT a cluster point of a_n" means

$\exists\varepsilon>0$, $\underbrace{\exists N\in\mathbb{N}, \text{ s.t. } \forall n>N}_{n>>1}$, $|a_n-K|\geq\varepsilon$

Step I: meaning of "a_n has one cluster point"

$\exists K\in\mathbb{R}$ s.t. $\forall\varepsilon>0$, $\forall N\in\mathbb{N}$, $\exists n>N$ s.t.

$|a_n-K|<\varepsilon$

"a_n has no cluster points" is

1) $\forall K\in\mathbb{R}$, $\exists\varepsilon>0$, $\exists N\in\mathbb{N}$, s.t. $\forall n>N$

$|a_n-K|\geq\varepsilon$

2) $\forall K\in\mathbb{R}$, $\exists\varepsilon>0$ s.t. $|a_n-K|\geq\varepsilon$, $n>>1$

Proof of " $a_n \to +\infty \Rightarrow \{a_n\}$ has no cluster points"

Step 0 & I above

Step II:

$\forall K \in \mathbb{R}$, let $\varepsilon = 1$. Then for $M = K+1$

$$a_n \to +\infty \Rightarrow a_n > M = K+1 \quad n \gg 1$$

$$\Rightarrow |a_n - K| > 1, \quad n \gg 1.$$

$a_n \to L$ means $\forall \varepsilon > 0$, $|a_n - L| < \varepsilon$, $n \gg 1$

Ex: True or false "$a_n \not\to L$" means $\exists \varepsilon > 0$, $|a_n - L| \geq \varepsilon$, for infinitely many n.

If $\{a_n\}$ does not converge to $L \in \mathbb{R}$, then there is some $\varepsilon > 0$ s.t. $|a_n - L| \geq \varepsilon$ for infinitely many n.

This is true because negation of "$|a_n - L| \geq \varepsilon$ for infinitely many n" is "$|a_n - L| < \varepsilon$, $n \gg 1$".

Ex: True or false

If $\{a_n\}$ has only one cluster point K, then $a_n \to K$

Counter example: $a_n = \begin{cases} 0, & n \text{ is even} \\ n, & n \text{ is odd} \end{cases}$

$\boxed{\text{Cluster point Theorem}}$

K is a cluster point of $\{a_n\}$ $\Longleftrightarrow$ K is a limit of some subsequence $\{a_{n_i}\}$

Proof: "$\Leftarrow$" is easy.

"$\Rightarrow$" Choose step by step a subsequence such that

$\forall \varepsilon > 0$, $|a_n - K| < \varepsilon$ for infinitely many n.

$$|a_{n_1} - K| < 1$$

$$|a_{n_2} - K| < \frac{1}{2}, \quad n_2 > n_1$$

$$|a_{n_3} - K| < \frac{1}{3}, \quad n_3 > n_2$$

$$\vdots$$

Prove that this is possible by induction.

① K is a cluster point

$\Rightarrow$ for $\varepsilon = 1$, there is at least one

n_1 s.t. $|a_{n_1} - K| < 1$.

② If we already find $n_1, \cdots, n_k$,

K is a cluster point

$\Rightarrow$ for $\varepsilon = \frac{1}{k+1}$, there are infinitely many n

s.t. $|a_n - K| < \frac{1}{k+1}$

$\Rightarrow$ there is one $n_{k+1} > n_k$ s.t.

$$|a_{n_{k+1}} - K| < \frac{1}{k+1}$$

So we have a subsequence $\{a_{n_k}\}$

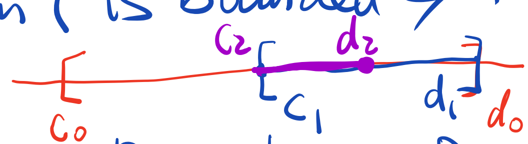
satisfying $0 \le |a_{n_k} - K| < \frac{1}{k}$

Squeeze Thm $\Rightarrow |a_{n_k} - K| \to 0$, $k \to \infty$.

Bolzano-Weierstrass Theorem

$\{a_n\}$ is bounded $\Rightarrow$ $\{a_n\}$ has a convergent subsequence.

Remark: $\{a_n\}$ is bounded $\Rightarrow$ $\{a_n\}$ has a cluster point.



Sketchy Proof: By Cluster Point Theorem, we only need to find one cluster point.

We construct it by Nested Interval Theorem.

1) $\{a_n\}$ is bounded $\Rightarrow$ $c_0 \leq a_n \leq d_0$, $\forall n$

2) Cut $[c_0, d_0]$ into two half-intervals

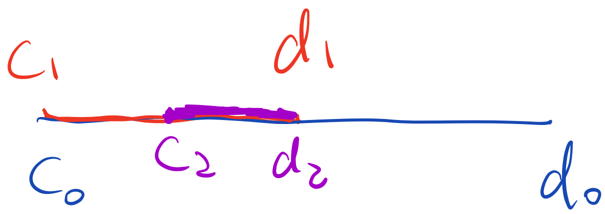
Let $[c_1, d_1]$ be a half containing infinitely many a_n

3) Cut $[c_1, d_1]$ into two half-intervals

Let $[c_2, d_2]$ be a half containing infinitely many a_n

4) Repeating this, we get nested intervals

$$[c_0, d_0] \supset [c_1, d_1] \supset [c_2, d_2] \supset \cdots$$



5) Let $b = d_0 - c_0$, then the n-th interval has length $\frac{1}{2^n} b \to 0$.

Nested Interval Thm $\Rightarrow$ $\lim_{n\to\infty} c_n = L = \lim_{n\to\infty} d_n$

6) L is the cluster point of $\{a_n\}$

because $\forall \varepsilon > 0$, $c_n \in (L-\varepsilon, L+\varepsilon)$, $n \gg 1$

$d_n \in (L-\varepsilon, L+\varepsilon)$, $n \gg 1$

$\Rightarrow$ $[c_n, d_n] \subset (L-\varepsilon, L+\varepsilon)$, $n \gg 1$

$\Rightarrow$ $(L-\varepsilon, L+\varepsilon)$ contains infinitely many a_n.

<u>Def</u> $\{a_n\}$ is a Cauchy sequence if

$\forall \varepsilon > 0$, $|a_m - a_n| < \varepsilon$, for $m, n >> 1$

$\forall \varepsilon > 0$, $\exists N_\varepsilon \in \mathbb{N}$, $\forall m, n > N_\varepsilon$, $|a_m - a_n| < \varepsilon$

<u>Theorem</u> $\{a_n\}$ is Cauchy $\Leftrightarrow$ $\{a_n\}$ converges

Proof: $\boxed{\text{"}\Leftarrow\text{"}}$ is HW#4 P1

$\boxed{\text{"}\Rightarrow\text{"}}$ ① $\{a_n\}$ is Cauchy $\Rightarrow$

For $\varepsilon = 1$, $\exists N$ s.t. $\forall m, n \geq N$,

$$|a_m - a_n| < \varepsilon = 1$$

$$\Rightarrow \quad |a_m - a_N| < 1, \quad \forall m \geq N$$

$$\Rightarrow 1 - a_N < a_m < a_N + 1, \quad \forall m \geq N$$

$\Rightarrow$ $\{a_n\}$ is bounded for $n >> 1$

$\Rightarrow$ $\{a_n\}$ is bounded

② Bolzano-Weierstrass Theorem

$\Rightarrow$ $\{a_n\}$ has a convergent subsequence

$\{a_{n_i}\}$

and $\lim_{i\to\infty} a_{n_i} = L$

③ $\forall \varepsilon > 0$, $\begin{cases} a_{n_i} \to L \Rightarrow |a_{n_i} - L| < \varepsilon, \ i \gg 1 \\ \{a_n\} \text{ is Cauchy} \Rightarrow |a_n - a_{n_i}| < \varepsilon, \quad n, n_i \gg 1 \end{cases}$

$\left.\begin{matrix} n_i \geq i \\ n, i \geq N \end{matrix}\right\} \Rightarrow n, n_i \geq N$

$$\Rightarrow |a_n - L| = |a_n - a_{n_i} + a_{n_i} - L|$$
$$\leq |a_n - a_{n_i}| + |a_{n_i} - L|$$
$$< \varepsilon + \varepsilon = 2\varepsilon, \quad n \gg 1$$

$\Rightarrow a_n \to L$.

Misunderstanding of Cauchy:

only $|a_{n+1} - a_n| < \varepsilon$, $n \gg 1$

Example: Fibonacci fraction defined by

$$\begin{cases} a_{n+1} = \dfrac{1}{a_n + 1}, & n \geq 1 \\ a_1 = 1 \end{cases}$$

$1, \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}, \frac{8}{13}, \cdots$

Prove the convergence by verifying Cauchy.

$$|a_n - a_{n+1}| = \left| \frac{1}{a_{n-1}+1} - \frac{1}{a_n+1} \right|$$

$$= \frac{|a_{n-1} - a_n|}{(a_n+1)(a_{n-1}+1)}$$

If $a_n \geq \frac{1}{2}$, $\forall n$, then $(a_n+1)(a_{n-1}+1) \geq \frac{3}{2} \cdot \frac{3}{2} > 2$

$\Rightarrow |a_n - a_{n+1}| < \frac{1}{2} |a_{n-1} - a_n|$, $\forall n$

$$\Rightarrow |a_n - a_{n+1}| < \frac{1}{2} |a_{n-1} - a_n|$$

$$< \frac{1}{2} \cdot \frac{1}{2} |a_{n-2} - a_{n-1}|$$

$$< \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} |a_{n-3} - a_{n-2}|$$

$$\cdots$$

$$< \frac{1}{2^{n-1}} |a_1 - a_2| = \frac{1}{2^n}$$

$m > n$

$$|a_n - a_m| = |a_n - a_{n+1} + a_{n+1} - a_{n+2} + a_{n+2} - \cdots + a_{m-1} - a_m|$$

$$\leq |a_n - a_{n+1}| + |a_{n+1} - a_{n+2}| + \cdots + |a_{m-1} - a_m|$$

$$\leq \frac{1}{2^n} + \frac{1}{2^{n+1}} + \cdots + \frac{1}{2^{m-1}}$$

$$= \frac{1}{2^n} \left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{m-1-n}} \right)$$

$$= \frac{1}{2^n} \frac{1 - \left(\frac{1}{2}\right)^{m-n}}{1 - \frac{1}{2}} < \frac{1}{2^n} \frac{1}{1 - \frac{1}{2}} = \frac{1}{2^{n-1}} < \varepsilon,$$

if $m > n > N_\varepsilon$.

To prove $\frac{1}{2} \le a_n \le 1$, $\forall n$ by induction:

① $a_1 = 1$

② $\frac{1}{2} \le a_n \le 1 \Rightarrow \frac{3}{2} \le a_n + 1 \le 2$

$\Rightarrow \frac{1}{2} \le \frac{1}{a_n + 1} \le \frac{2}{3}$

$\Rightarrow \frac{1}{2} \le a_{n+1} \le \frac{2}{3} < 1$.