Review

Def $\{a_n\}$ converges if $a_n \to L$ for some $L \in \mathbb{R}$

$\exists L \in \mathbb{R}$ s.t. ① $a_n \to L$

② $\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n > N, |a_n - L| < \varepsilon$

$\{a_n\}$ diverges if

$\forall L \in \mathbb{R}, \exists \varepsilon > 0$, s.t. $\underbrace{\forall N \in \mathbb{N}, \exists n > N, |a_n - L| \geq \varepsilon}_{|a_n - L| \geq \varepsilon \text{ for infinitely many } n}$

Remark: Negation of "$n \gg 1$" is "for infinitely many n".
And vice versa.

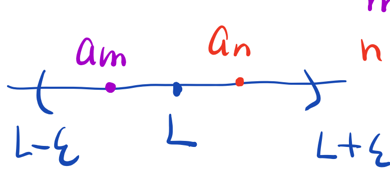
Def $\{a_n\}$ is Cauchy if $\forall \varepsilon > 0, |a_m - a_n| < \varepsilon$, $m \gg 1$, $n \gg 1$.

$\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $|a_m - a_n| < \varepsilon$, $\forall m > N$, $\forall n > N$.

$\boxed{\text{Theorem}}$ $\{a_n\}$ is Cauchy $\iff$ $\{a_n\}$ converges

HW#4 P1 Prove "$\Leftarrow$"

Intuition:



$m \gg 1$
$n \gg 1$

Negation of "$\{a_n\}$ is Cauchy"

$\exists \varepsilon > 0, \forall N \in \mathbb{N}, \exists m, n > N$ s.t. $|a_m - a_n| \geq \varepsilon$

Exercise: Contrapositive of "$\Leftarrow$" above.

Def $K \in \mathbb{R}$ is a cluster point of $\{a_n\}$ if

$\forall \varepsilon > 0$, $|a_n - K| < \varepsilon$ for infinitely many n

$\forall \varepsilon > 0$, $\forall N \in \mathbb{N}$, $\exists n > N$, s.t. $|a_n - K| < \varepsilon$

n depends on ε and N.

K is not a cluster point of $\{a_n\}$:

$\exists \varepsilon > 0$, $\underbrace{\exists N \in \mathbb{N}, \text{ s.t. } \forall n > N}_{n >> 1}$, $|a_n - K| \geq \varepsilon$

Cluster Point Theorem

K is a cluster point of $\{a_n\}$

$\Longleftrightarrow$ K is the limit of some subsequence $\{a_{n_i}\}$

Bolzano-Weierstrass Theorem

$\{a_n\}$ is bounded $\Rightarrow$ $\{a_n\}$ has at least one convergent subsequence

Remark : $\{a_n\}$ is bounded $\Rightarrow$ $\{a_n\}$ has at least one cluster point

Completeness Theorem $\qquad (-1)^n \frac{1}{n}$

$\{a_n\}$ is bounded and monotone $\Rightarrow$ $\{a_n\}$ converges

Ex: True or False

F ① $\{a_n\}$ diverges $\Rightarrow$ $\{a_n\}$ has no cluster points $\quad (-1)^n$

T ② $\{a_n\}$ has no cluster points $\Rightarrow$ $\{a_n\}$ is unbounded

T ③ $\{a_n\}$ has no cluster points $\Rightarrow$ $\{a_n\}$ diverges

F ④ $\{a_n\}$ has one and only one cluster point K
$\Rightarrow a_n \to K$ $\quad a_n = \begin{cases} 1, & n \text{ is odd} \\ n, & \text{even} \end{cases}$

F ⑤ $\{a_n\}$ diverges $\Rightarrow$ $\{a_n\}$ is unbounded $\quad (-1)^n$

F ⑥ If all cluster points of $\{a_n\}$ are the same $\Rightarrow$ $\{a_n\}$ converges.

F ⑦ $\{a_n\}$ converges $\Rightarrow$ it's bounded and monotone
$(-1)^n \cdot \frac{1}{n}$

Characterization of Convergence / Limit

The following are sufficient for convergence

① $a_n \to L$ for some $L \in \mathbb{R}$

② $\{a_n\}$ is Cauchy

③ $\{a_n\}$ is bounded and monotone

The following are sufficient conditions of a convergent subsequence:

1) $\{a_n\}$ is bounded

2) $\{a_n\}$ has a cluster point

To prove $\{a_n\}$ converges, we can

(a) Verify $a_n \to L$ by definition

(b) If L is unknown, verify it's Cauchy

(c) Show it's bounded and monotone

(d) Find $c_n \to L$, $b_n \to L$ s.t.
$c_n \le a_n \le b_n$

$\boxed{\text{Sets}}$

<u>Def</u> Let $S \subseteq \mathbb{R}$ be a set of real numbers.

① An upper bound of S is a number b s.t. $x \le b$, $\forall x \in S$.

② S is bounded above if S has an upper bound.

③ m is the maximum if m is one upper bound of S and $m \in S$

denoted as $m = \max S$

Ex: $S = [0,1]$, $\max S = 1$, $\sup S = 1$

④ The supremum of S ($\sup S$) is a number $\bar{m}$ s.t. $\bar{m}$ is the smallest upper bound of S

Ex: 1) $S = [0,1)$, $\sup S = 1$, $\max S$ DNE

2) $S = \{1 - \frac{1}{n} : n \in \mathbb{N}\}$ $0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}$

$\sup S = 1$, $\max S$ does not exist

3) $S = \{1 + \frac{1}{n} : n \in \mathbb{N}\}$ $2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \ldots$

$\sup S = 2$, $\max S = 2$

4) $S = \mathbb{N}$ sup and max do not exist.

$\boxed{\text{Theorem}}$ ① If $\max S$ exists, then $\sup S = \max S$.

② max and sup are unique if they exist

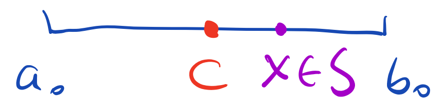
lower bound, minimum, infinimum (inf) are similarly defined.

Theorem (Completeness for sets of real numbers)

If S is non-empty and bounded above, $\sup S$ exists.

Proof: S is non-empty $\Rightarrow \exists\, a_0 \in S$

bounded above $\Rightarrow \exists\, b_0$ s.t. $x \le b_0,\ \forall x \in S$



Bisect $[a_0, b_0]$, pick one half and call it $[a_1, b_1]$

s.t. b_1 is an upper bound of S and $[a_1, b_1]$ contains one point in S:

1) If c is upper bound, then $[a_1, b_1]$ is left half

2) If not, then right half

Repeat the process, we get nested intervals:

$$[a_0, b_0] \supseteq [a_1, b_1] \supseteq \cdots \supseteq [a_n, b_n] \supseteq \cdots$$

Nested Interval Theorem $\Rightarrow \lim a_n = L = \lim b_n$

Claim $L = \sup S$:

① L is an upper bound

for any fixed $x \in S$, b_n is upper bound

$\Rightarrow x \le b_n,\ \forall n \Rightarrow x \le \lim b_n$

Limit Location Thm

② L is the smallest upper bound:

If M is one upper bound of S, then

$[a_n, b_n]$ contains one point in S

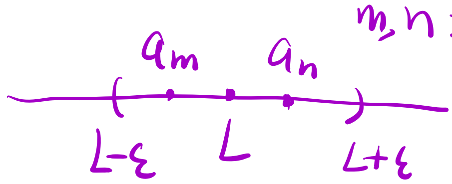
$\Rightarrow a_n \le M, \forall n$

$\Rightarrow \lim a_n \le M$

HW # 4

$m, n > N$

$\exists L, \ a_n \to L$

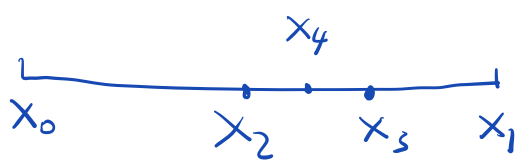


1. (20 pts) Prove that every convergent sequence is a Cauchy sequence.

2. (20 pts) Page 91, Problem 6-1.
   The sequence x_n is defined by $x_0 = a, x_1 = b$ and recursive relation

$$x_n = \frac{x_{n-1} + x_{n-2}}{2}, \quad n \ge 2.$$

   (a) Prove that $\{x_n\}$ is Cauchy.



   (b) Find $\lim x_n$ in terms of a and b.

(a)

$$x_{n+1} - x_n = \frac{x_n + x_{n-1}}{2} - x_n = \frac{x_{n-1} - x_n}{2} \Rightarrow |x_{n+1} - x_n| = \frac{1}{2}|x_n - x_{n-1}|$$

$$\Rightarrow |x_{n+1} - x_n| = \frac{1}{2}|x_n - x_{n-1}| = \frac{1}{2^2}|x_{n-1} - x_{n-2}| = \cdots = \frac{1}{2^{n-1}}(b-a)$$

$$|x_m - x_n| = |x_m - x_{m-1} + x_{m-1} - x_{m-2} + x_{m-2} + \cdots + x_{n+1} - x_n|$$

(b)

$x_0 = a$

$x_1 = b$

$x_2 = a + \frac{b-a}{2}$

$x_3 = a + \frac{b-a}{2} + \frac{b-a}{4}$

$x_4 = a + \frac{b-a}{2} + \frac{b-a}{4} - \frac{b-a}{8}$

$(-\frac{1}{2})^n (b-a)$

$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots + (-\frac{1}{2})^n$

3. (10 pts) Page 90, Exercise 6.5: 1(b)(d).
For the following two sets, determine the sup, inf, max, min if they exist:

$= \frac{1-(-\frac{1}{2})^{n+1}}{1-(-\frac{1}{2})}$

(a) $\{[\cos(n\pi)]/n : n \in \mathbb{N}\}$.

(b) $\{n2^{-n} : n \in \mathbb{N}\}$. $\frac{1}{2}, \frac{2}{4}, \frac{3}{8}, \ldots$

$\frac{n}{2^n} \to 0$

4(a) $A \subseteq B \Rightarrow \sup A \leq \sup B$

(g) $\sup(A+B) \leq \sup A + \sup B$

5. (20 pts) Page 91, Problem 6-3.
$f(x)$ is continuous and decreasing on $[0, \infty]$ and $f(n) \to 0$. For

$$a_n = f(0) + f(1) + \cdots + f(n-1) - \int_0^n f(x)dx,$$

(a) Prove a_n is Cauchy.

(b) For $f(x) = e^{-x}$, find the limit of a_n.

6. (20 pts) Page 90, Exercise 6.4: 2.
Suppose a_n has this property: there is C and K with $0 < K < 1$ s.t.

$$|a_n - a_{n+1}| < CK^n, \quad n \gg 1.$$

Prove that a_n is Cauchy.

5(a) $a_{n+1} - a_n = f(n) - \int_0^{n+1} f(x)dx + \int_0^n f(x)dx$

$= f(n) - \int_n^{n+1} f(x)dx$