Homework 2

Due before 10am on September 8th on gradescope.

1. (20 pts) Page 47, 3.4/3. Show by definition that if $a > 1$, $a^n/n \longrightarrow +\infty$.

   Hint: use Binomial Theorem.

2. (20 pts) Page 47, 3.6/1. Show that

$$\int_1^2 \ln^n x\, dx \longrightarrow 0$$

   and

$$\int_2^3 \ln^n x\, dx \longrightarrow +\infty$$

   Hint: $\ln 2 = 0.69...$ and $e = 2.71...$

3. (20 pts) Page 47, 3.4/5. Prove that $a^{\frac{1}{n}} \longrightarrow 1$ if $a > 0$.
   Hint: Following the hint in the book: for the case $a > 1$, we know $a^{\frac{1}{n}} > 1$ thus $a^{\frac{1}{n}} = 1 + h_n$ for some $h_n > 0$; then by Binomial Theorem

$$a = (1 + h_n)^n = 1 + nh_n + \frac{1}{2}n(n-1)h_n^2 + \cdots + h_n^n.$$

   Try to derive an inequality from the equation above so that you can show $h_n \to 0$ by definition.

4. (20 pts) Page 59, Problem 4-1. Prove that $n^{\frac{1}{n}} \to 1$.
   Hint: Let $e_n = n^{\frac{1}{n}} - 1$, then by $n^{\frac{1}{n}} = e_n + 1$ and Binomial Theorem, we get

$$n = (1 + e_n)^n = 1 + ne_n + \frac{1}{2}n(n-1)e_n^2 + \cdots + e_n^n.$$

   We know $e_n > 0$. Try to derive an inequality from the equation above so that you can show $e_n \to 0$ by definition.

5. (20 pts) Page 48, Problem 3-1.
   For a given sequence $\{a_n\}$, another sequence $\{b_n\}$ is defined as its average:

$$b_n = \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

   (a) Prove that if $a_n \to 0$, then $b_n \to 0$.

(b) Deduce from part (a) in a few lines that if $a_n \to L$, then $b_n \longrightarrow L$.

Hint: $a_n \to 0$ means that for any fixed $\epsilon > 0$, there is an N s.t. $|a_n| < \epsilon$ for any $n \geq N$. Thus $\left|\frac{a_{N+1}+a_{N+2}+\cdots+a_n}{n}\right|$ is smaller than ϵ. Show that if n is large enough (find that index) then the other part of b_n is also smaller than ϵ (this is possible because N is fixed for fixed ϵ).