Homework 4

Due on Sep 22 before 10am on Gradescope.

1. (20 pts) Prove that every convergent sequence is a Cauchy sequence.

2. (20 pts) Page 91, Problem 6-1.
   The sequence x_n is defined by $x_0 = a, x_1 = b$ and recursive relation

   $$x_n = \frac{x_{n-1} + x_{n-2}}{2}, \quad n \geq 2.$$

   (a) Prove that $\{x_n\}$ is Cauchy.

   (b) Find $\lim x_n$ in terms of a and b.

3. (10 pts) Page 90, Exercise 6.5: 1(b)(d).
   For the following two sets, determine the sup, inf, max, min if they exist:
   (a) $\{[\cos(n\pi)]/n : n \in \mathbb{N}\}$.

   (b) $\{n2^{-n} : n \in \mathbb{N}\}$.

4. (10 pts) Page 90, Exercise 6.5: 3(a)(g).

5. (20 pts) Page 91, Problem 6-3.
   $f(x)$ is continuous and decreasing on $[0, \infty]$ and $f(n) \to 0$. For

   $$a_n = f(0) + f(1) + \cdots + f(n-1) - \int_0^n f(x)dx,$$

   (a) Prove a_n is Cauchy.

   (b) For $f(x) = e^{-x}$, find the limit of a_n.

6. (20 pts) Page 90, Exercise 6.4: 2.
   Suppose a_n has this property: there is C and K with $0 < K < 1$ s.t.

   $$|a_n - a_{n+1}| < CK^n, \quad n \gg 1.$$

   Prove that a_n is Cauchy.