Homework 8

Due on Oct 27th before 10am on gradescope.

1. (30 pts) Consider the Maclaurin seris for $\sin x$:

$$\sin x = \sum_{n=1}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

   (a) (10 pts) Prove that the series $\sum_{n=1}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$ converges for any $x \in [0,1]$. Thus $\sum_{n=1}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$ also converges for any $x \in [-1,0]$ since the only difference is a sign.

   (b) (15 pts) Prove that $|\sin x - x| \leq \frac{|x|^3}{3!}$ for any $x \in [-1,1]$. Hint: follow the proof of Alternating Series Test Theorem.

   (c) (5 pts) Use the estimate above to show $|x| < 0.1 \Rightarrow |\sin x - x| < 0.001$.

2. (10 pts) Prove that $\sum_{n=1}^{N} a_n \cos(nx)$ is bounded on $(-\infty, +\infty)$.

3. (10 pts) Show that $\int_0^1 \frac{x^4}{1+x^6} dx \leq \frac{1}{5}$ by estimating the integrand.

4. (10 pts) For what values of $k > 0$ are the function $f(x)$ bounded for $x \approx 0+$?

   (a) $f(x) = \int_x^1 (1/t^k) dt$.

   (b) $f(x) = \int_x^1 (e^t/t^k) dt$.

5. (10 pts) Show that a function which is locally increasing on an interval I is increasing on I. Hint: try an indirect argument (or proof by contradiction) and use bisection to construct nested intervals.

6. (10 pts) If $f(x)$ is continuous at x_0, show $f(x)$ is locally bounded at x_0.

7. (20 pts) P167, Exercise 11.3/1.