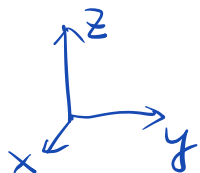


Review :

- A Vector Space $V = \mathbb{R}^3$ (or $V = \mathbb{R}^{1 \times 3}$) can be perceived as:



- We call W a subspace of V if $W \subseteq V$ and W is also a vector space.

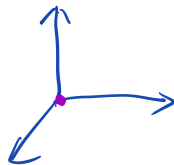
- A vector space or a subspace is **closed under two operations.**

$$\vec{0} = 0 \cdot \vec{v} \in W$$

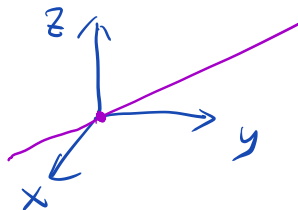
- $\vec{0}$ is always included in a subspace.

- All possible subspaces of \mathbb{R}^3 are :

1) $W = \{ \vec{0} \} = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

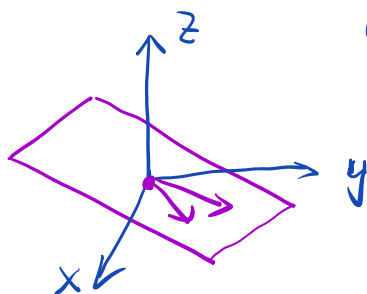


2) $W = \text{Span} \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \right\}$ where $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.



3) $W = \text{Span} \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \begin{bmatrix} d \\ e \\ f \end{bmatrix} \right\}$ where $\begin{bmatrix} a \\ b \\ c \end{bmatrix}, \begin{bmatrix} d \\ e \\ f \end{bmatrix}$ are not

on the same line.



$$4) W = \text{Span} \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \begin{bmatrix} d \\ e \\ f \end{bmatrix}, \begin{bmatrix} g \\ h \\ i \end{bmatrix} \right\} = \mathbb{R}^3$$

$\downarrow \quad \downarrow \quad \downarrow$
 not on the same plane.

- The column space of a matrix A is the subspace **spanned by all column vectors of A** .

Example: ① $A = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}$

$$\text{Col}(A) = \text{Span} \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \begin{bmatrix} d \\ e \\ f \end{bmatrix}, \begin{bmatrix} g \\ h \\ i \end{bmatrix} \right\} \subseteq \mathbb{R}^3$$

$$\textcircled{2} A = \begin{bmatrix} | & | & | \\ | & | & | \\ | & | & | \end{bmatrix} \Rightarrow \text{Col}(A) = \text{Span} \left\{ \begin{bmatrix} | \\ | \\ | \end{bmatrix} \right\}$$

$$\text{Span} \left\{ \begin{bmatrix} | \\ | \\ | \end{bmatrix}, \begin{bmatrix} | \\ | \\ | \end{bmatrix}, \begin{bmatrix} | \\ | \\ | \end{bmatrix} \right\} = \text{Span} \left\{ \begin{bmatrix} | \\ | \\ | \end{bmatrix} \right\}$$

$$\textcircled{3} A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{Col}(A) = \text{Span} \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

- The null space of A is the set of all solutions to the homogeneous linear system $A\vec{x} = \vec{0}$ and it is a subspace of \mathbb{R}^n if $A \in \mathbb{R}^{m \times n}$

$A \cdot \vec{x} = \vec{b}$

$$\text{Col}(A) \subseteq \mathbb{R}^m$$

$$\text{Null}(A) \subseteq \mathbb{R}^n$$

Example: ① $A = \begin{pmatrix} 1 & 3 & 2 \\ 1 & 2 & 3 \\ 0 & 1 & -1 \end{pmatrix}$ $\text{Null}(A)$
 $= \{ \text{all sols to } A\vec{x} = \vec{0} \}$

The RREF of $[A | \vec{0}]$ is $\begin{pmatrix} 1 & 3 & 2 \\ 1 & 2 & 3 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 1 & 0 & 5 & | & 0 \\ 0 & 1 & -1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

$$z = t \Rightarrow \begin{cases} y = z = t \\ x = -5z = -5t \end{cases}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -5t \\ t \\ t \end{pmatrix} = t \begin{pmatrix} -5 \\ 1 \\ 1 \end{pmatrix}, \quad \forall t \in \mathbb{R}.$$

$$\Rightarrow \text{Null}(A) = \text{Span} \left\{ \begin{bmatrix} -5 \\ 1 \\ 1 \end{bmatrix} \right\} = \left\{ t \begin{pmatrix} -5 \\ 1 \\ 1 \end{pmatrix}, \forall t \in \mathbb{R} \right\}$$

② All sols to $A\vec{x} = \vec{b}$ for

$$\begin{pmatrix} 1 & 3 & 2 & | & 1 \\ 1 & 2 & 3 & | & 0 \\ 0 & 1 & -1 & | & 1 \end{pmatrix}$$

RREF of $[A | \vec{b}]$ is

$$\begin{pmatrix} 1 & 0 & 5 & | & -2 \\ 0 & 1 & -1 & | & 1 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

$$z = t \Rightarrow \begin{cases} y = 1 + z = 1 + t \\ x = -2 - 5z = -2 - 5t \end{cases}$$

$$\begin{aligned} & \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + \text{Span} \left\{ \begin{bmatrix} -5 \\ 1 \\ 1 \end{bmatrix} \right\} \\ & = \left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -5 \\ 1 \\ 1 \end{pmatrix}, \forall t \in \mathbb{R} \right\} \end{aligned}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 - 5t \\ 1 + t \\ t \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -5t \\ t \\ t \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + t \underbrace{\begin{pmatrix} -5 \\ 1 \\ 1 \end{pmatrix}}_{\text{Null}(A)}, \quad \forall t \in \mathbb{R}.$$

The set of sols to $A\vec{x}=\vec{b}$ is equal to $\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + \text{Span}\left\{\begin{bmatrix} -5 \\ 1 \\ 1 \end{bmatrix}\right\}$,
if there are solutions Null(A).

- Fact: all sols to $A\vec{x}=\vec{b}$ can be written as

$$\vec{x}_p + \text{Null}(A)$$

where \vec{x}_p is a sol to $A\vec{x}=\vec{b}$.

-
- Definition (Linear Independence)

A set of (abstract) vectors $S = \{\vec{v}_1, \dots, \vec{v}_n\}$ is called linearly dependent if there are scalars a_1, \dots, a_n which are not all zeros,

st. $a_1\vec{v}_1 + a_2\vec{v}_2 + \dots + a_n\vec{v}_n = \vec{0}$.

Otherwise, S is linearly independent.

Remark: As long as one of a_i is not zero, it satisfies the definition

Examples: Assume $2\begin{bmatrix} a \\ b \\ c \end{bmatrix} + \begin{bmatrix} d \\ e \\ f \end{bmatrix} - \begin{bmatrix} g \\ h \\ i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$,

then they are linearly dependent, and

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} d \\ e \\ f \end{bmatrix} + \frac{1}{2}\begin{bmatrix} g \\ h \\ i \end{bmatrix} \in \text{Span}\left\{\begin{bmatrix} d \\ e \\ f \end{bmatrix}, \begin{bmatrix} g \\ h \\ i \end{bmatrix}\right\}$$

$$\Rightarrow \text{Span} \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \begin{bmatrix} d \\ e \\ f \end{bmatrix}, \begin{bmatrix} g \\ h \\ i \end{bmatrix} \right\} = \text{Span} \left\{ \begin{bmatrix} d \\ e \\ f \end{bmatrix}, \begin{bmatrix} g \\ h \\ i \end{bmatrix} \right\}$$

So dependency means redundancy: at least one vector can be spanned by others.

- Two vectors in \mathbb{R}^3 are dependent means they are on the same line:

assume $\vec{u}, \vec{v} \in \mathbb{R}^3$ are dependent
then there are numbers a, b s.t.

$$a\vec{u} + b\vec{v} = \vec{0}$$

and at least one of a, b is not 0.

Without loss of generality, let $a \neq 0$, then

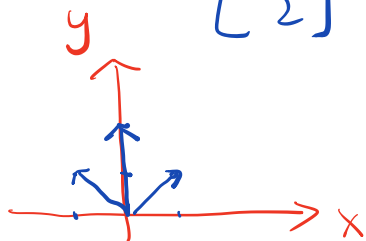
$$a\vec{u} = -b\vec{v}$$

$$\Rightarrow \vec{u} = -\frac{b}{a}\vec{v}$$

$$\Rightarrow \vec{u} \parallel \vec{v}$$

- Example: $\left\{ \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ are dependent

because $-\begin{bmatrix} 0 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$



Examples: $S = \{\vec{0}\}$ is dependent.

① If S contains $\vec{0}$, then S is dependent.

$$S = \{\vec{0}, \vec{u}, \vec{v}, \vec{w}\} \Rightarrow 2 \cdot \vec{0} + 0 \cdot \vec{u} + 0 \cdot \vec{v} + 0 \cdot \vec{w} = \vec{0}$$

② Check whether cols of $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 5 \\ 1 & 0 & 3 \end{bmatrix}$ are

independent: assume $a \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 3 \\ 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

then $\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 5 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 2 & 1 & 5 & 0 \\ 1 & 0 & 3 & 0 \end{array} \right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

there are nonzero solutions

\Rightarrow dependent.

③ Check whether rows of A are dependent:

$$a \begin{bmatrix} 1 & 0 & 3 \end{bmatrix} + b \begin{bmatrix} 2 & 1 & 5 \end{bmatrix} + c \begin{bmatrix} 1 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} a + 2b + c = 0 \\ b = 0 \\ 3a + 5b + 3c = 0 \end{cases}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 3 & 5 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

④ Check whether $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ is independent.

$$\text{Sol: } a \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

equate each entry in the matrix \Rightarrow

$$\begin{cases} a+b=0 \\ a+c=0 \\ \boxed{0=0} \\ a+b=0 \end{cases}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}_{4 \times 3} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

⑤ Check whether $\{1+x^2, 1-x+x^2, 1\}$ is independent

$$\text{Let } a(1+x^2) + b(1-x+x^2) + c \cdot 1 = 0$$

$$\Rightarrow (a+b+c) + (-b) \cdot x + (a+b)x^2 = 0$$

$$\Rightarrow \begin{cases} a+b=0 \\ -b=0 \\ a+b+c=0 \end{cases}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$


$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{array} \right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right] \Rightarrow \text{only zero sol}$$

\Rightarrow independent.

• Definition (Basis & Dimension)

If a vector space V is spanned by
some linearly independent vectors,
then these vectors are called a basis

of V . The number of basis vectors is called the dimension of V . 

Remark: For a vector space, there could be many different bases.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Examples:

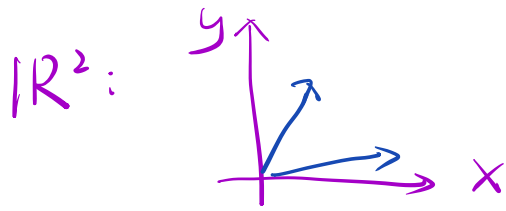
① $V = \mathbb{R}^3$, $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis.

② $V = \mathbb{R}^3$, $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ is also a basis.

③ $V = \mathbb{R}^{2 \times 2}$, $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is a basis

$$ax^2 + bx + c = a \cdot x^2 + b \cdot x + c \cdot 1$$

④ $V = P_2(\mathbb{R}) = \text{Span} \{ 1, x, x^2 \}$



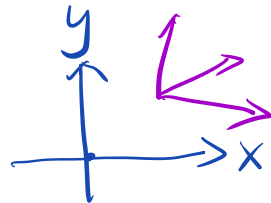
Theorem For any vector space V ,

① different bases have the same number of vectors

② the maximum number of linearly independent vectors is the dimension of V .

③ the minimum number of the vectors spanning V is the dimension of V .

Example: ① $V = \mathbb{R}^2$



$$V = \text{Span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\} \Rightarrow \dim(V) = 2.$$

Three vectors in \mathbb{R}^2 are always dependent.

$$\textcircled{2} \quad V = \mathbb{R}^3$$

$$x \begin{bmatrix} a \\ b \\ c \end{bmatrix} + y \begin{bmatrix} d \\ e \\ f \end{bmatrix} + z \begin{bmatrix} g \\ h \\ i \end{bmatrix} + w \begin{bmatrix} j \\ k \\ l \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} a & d & g & j \\ b & e & h & k \\ c & f & i & l \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} a & d & g & j & 0 \\ b & e & h & k & 0 \\ c & f & i & l & 0 \end{array} \right]$$

Three rows \Rightarrow at most 3 pivots in RREF

\Rightarrow at least one col without pivot

\Rightarrow at least one free parameter

\Rightarrow nonzero sol for $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$

\Rightarrow dependent