Review

Def For $A \in \mathbb{R}^{m\times n}$, $\begin{bmatrix} a & b & c & d \\ e & f & g & h \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $Col(A) \subseteq \mathbb{R}^2$, $Row(A) \subseteq \mathbb{R}^{1\times 4}$, $Null(A) \subseteq \mathbb{R}^4$

① $\dim(Col(A))$ is called col rank of A

② $\dim(Row(A))$ is called row rank of A

③ $\dim(Null(A))$ is called nullity of A.

Theorem: ① Number of pivots in RREF(A) or RREF($[A \mid \vec{0}]$) is equal to col rank of A

② Number of pivots in RREF(A) or RREF($[A \mid \vec{0}]$) is equal to row rank of A

$\boxed{\text{Def}}$ The rank of A is defined as number of pivots in RREF(A).

$\begin{bmatrix} a & e \\ b & f \\ c & g \\ d & h \end{bmatrix}$

Assume $A \in \mathbb{R}^{m\times n}$, let r be its rank.
Then RREF$[A \mid \vec{0}]$ has r leading ones.

How to find a basis for Null(A), Col(A), Row(A)

① For Null(A), solve $A\vec{x}=\vec{0}$ and write Null(A) as a span of $(n-r)$ vectors, which are the basis vectors.

Example: $A=[1\ \ 1\ \ 1]$

$$\text{RREF}[A|\vec{0}]=[1\ \ \boxed{1}\ \ \boxed{1}\ |\ 0]$$

$$\Rightarrow \begin{cases} y=s \\ z=t \end{cases} \Rightarrow x=-s-t$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} -s-t \\ s \\ t \end{pmatrix}=\begin{pmatrix} -s \\ s \\ 0 \end{pmatrix}+\begin{pmatrix} -t \\ 0 \\ t \end{pmatrix}$$

$$= s\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}+t\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \quad \forall s,t\in\mathbb{R}$$

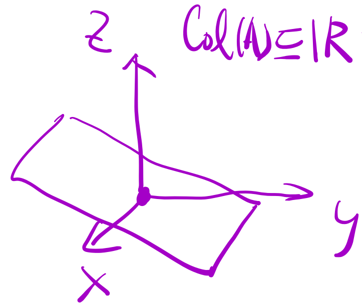
$$\Rightarrow \text{Null}(A)=\text{Span}\left\{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix},\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}$$

$\Rightarrow \left\{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix},\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}$ is a basis for Null(A)

② Cols in A corresponding to pivots in RREF form a basis for Col (A).

Example: $A = \begin{pmatrix} 1 & 3 & 2 \\ 1 & 2 & 3 \\ 0 & 1 & -1 \end{pmatrix}$

$\mathrm{Col}(A) \subseteq \mathbb{R}^3$



$a\begin{bmatrix}1\\1\\0\end{bmatrix} + b\begin{bmatrix}3\\2\\1\end{bmatrix} + c\begin{bmatrix}2\\3\\-1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

The RREF of $[A \mid \vec{0}]$:

$$\begin{bmatrix} 1 & 3 & 2 \\ 1 & 2 & 3 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \quad \left(\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$\mathrm{Col}(A) = \mathrm{Span}\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}3\\2\\1\end{bmatrix}, \begin{bmatrix}2\\3\\-1\end{bmatrix}\right\}$

$\Rightarrow \left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}3\\2\\1\end{bmatrix}\right\}$ is a basis for Col (A)

Remark: $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$ cannot be the basis because they cannot span $\begin{bmatrix}3\\2\\1\end{bmatrix}$.

③ Basis for Row (A): three methods

1) rows with pivots in RREF is a basis

Example: $A = \begin{pmatrix} 1 & 3 & 2 \\ 1 & 2 & 3 \\ 0 & 1 & -1 \end{pmatrix}$

The RREF of $[A \mid \vec{0}]$: $\mathrm{RREF} = E_m \cdots E_1 A$

$$\left(\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$A = E_1^{-1} E_2^{-1} \cdots E_m^{-1} \cdot RREF$

one row operation is a linear comb. of rows

$\Rightarrow \{[1\ 0\ 5], [0\ 1\ -1]\}$ is a basis

2) the rows in A ~~corresponding to pivots~~

(be careful that rows might be switched during Gaussian Elimination)

So avoid using this method unless you're sure.

Example: $\{[1\ 3\ 2], [1\ 2\ 3]\}$ is a basis.

3) Treat rows of A as abstract vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$, then solve for linear independence: $a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_3 = \vec{0}$

I. Independent $\Rightarrow \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis.

II. {The abstract vectors corresponding to colums with pivots in RREF} form a basis.

Notice that this method applies to any abstract vector (e.g. col vectors).

Example: $A = \begin{pmatrix} 1 & 3 & 2 \\ 1 & 2 & 3 \\ 0 & 1 & -1 \end{pmatrix}$

Basis for Row(A).

$$\begin{cases} a + b + 0\cdot c = 0 \\ 3a + 2b + c = 0 \\ 2a + 3b - c = 0 \end{cases}$$

$$a[1\ 3\ 2] + b[1\ 2\ 3] + c[0\ 1\ -1] = [0\ 0\ 0]$$

$$\Rightarrow [a\ b\ c]\begin{bmatrix} 1 & 3 & 2 \\ 1 & 2 & 3 \\ 0 & 1 & -1 \end{bmatrix} = [0\ 0\ 0]$$

$(AB)^T = B^T A^T$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 3 & 2 & 1 \\ 2 & 3 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 2 & 3 & -1 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & \boxed{1} & 0 \\ 0 & 1 & \boxed{-1} & 0 \\ 0 & 0 & \boxed{0} & 0 \end{array}\right]$$

So $\{[1\ 3\ 2], [1\ 2\ 3]\}$ is a basis of Row(A).

Example: Find a basis for

$\text{Span}\{1+3x+2x^2,\ 1+2x+3x^2,\ x-x^2\} \subseteq P_2(\mathbb{R})$

$a(1+3x+2x^2)+b(1+2x+3x^2)+c(x-x^2)$

$(a+b)+(3a+2b+c)x+(2a+3b-c)x^2=0$

$$\Rightarrow \begin{cases} a+b+0\cdot c=0 \\ 3a+2b+c=0 \\ 2a+3b-c=0 \end{cases}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 3 & 2 & 1 \\ 2 & 3 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 2 & 3 & -1 & 0 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

2nd poly 3rd poly

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So $\{1+3x+2x^2,\ 1+2x+3x^2\}$ is a basis.

Ex: Find a basis for

$$\text{Span}\left\{\begin{bmatrix}1&2\\3&0\end{bmatrix}, \begin{bmatrix}1&1\\1&1\end{bmatrix}, \begin{bmatrix}0&1\\1&1\end{bmatrix}\right\} \subseteq \mathbb{R}^{2\times 2}$$

Solution: Step I: check linear independence

$$a\begin{bmatrix}1&2\\3&0\end{bmatrix} + b\begin{bmatrix}1&1\\1&1\end{bmatrix} + c\begin{bmatrix}0&1\\1&1\end{bmatrix} = \begin{bmatrix}0&0\\0&0\end{bmatrix}$$

$$\begin{bmatrix}a&2a\\3a&0\end{bmatrix} + \begin{bmatrix}b&b\\b&b\end{bmatrix} + \begin{bmatrix}0&c\\c&c\end{bmatrix} = \begin{bmatrix}0&0\\0&0\end{bmatrix}$$

$$\begin{bmatrix}a+b & 2a+b+c\\3a+b+c & b+c\end{bmatrix} = \begin{bmatrix}0&0\\0&0\end{bmatrix}$$

$$\begin{cases} a+b+0\cdot c = 0\\ 2a+b+c=0\\ 3a+b+c=0\\ 0\cdot a+b+c=0\end{cases}$$

Step II: Augmented matrix, then RREF/REF

$$\left[\begin{array}{ccc|c}1&1&0&0\\2&1&1&0\\3&1&1&0\\0&1&1&0\end{array}\right]$$

$$\to \left[\begin{array}{ccc|c}1&1&0&0\\0&-1&1&0\\0&-2&1&0\\0&1&1&0\end{array}\right] \to \left[\begin{array}{ccc|c}1&1&0&0\\0&1&-1&0\\0&0&-1&0\\0&0&2&0\end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \text{ is REF, not RREF}$$

Step III :

$$\left\{ \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \right\} \text{ is a basis}$$

If RREF is $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$, then

$$\left\{ \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \right\} \text{ is a basis.}$$