**Def** For $A \in \mathbb{R}^{m\times n}$

① $\dim(\mathrm{Col}(A))$ is called col rank of $A$

② $\dim(\mathrm{Row}(A))$ is called row rank of $A$

③ $\dim(\mathrm{Null}(A))$ is called nullity of $A$.

Theorem: ① Number of pivots in RREF(A) or RREF($[A \mid \vec{0}]$) is equal to col rank of $A$

② Number of pivots in RREF(A) or RREF($[A \mid \vec{0}]$) is equal to row rank of $A$

$\boxed{\text{Def}}$ The rank of $A$ is defined as number of pivots in RREF(A)

Dimension Theorem: $A \in \mathbb{R}^{m\times n}$,

$$\operatorname{rank}(A) + \operatorname{nullity}(A) = n$$

Proof: $\operatorname{rank}(A)$ is # of pivots

$\operatorname{nullity}(A)$ is # of free parameters.

Example: $A = \begin{pmatrix} 1 & 3 & 2 \\ 1 & 2 & 3 \\ 0 & 1 & -1 \end{pmatrix}$ RREF $= \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$

$$\Rightarrow \begin{cases} \text{rank}(A) = 2 \\ \text{nullity}(A) = 1 \end{cases}$$

Geometric Meaning:

$\Rightarrow$ Col(A) is a 2-dimensional subspace of $\mathbb{R}^3$

Row(A) is a 2-dimensional subspace of $\mathbb{R}^{1\times 3}$

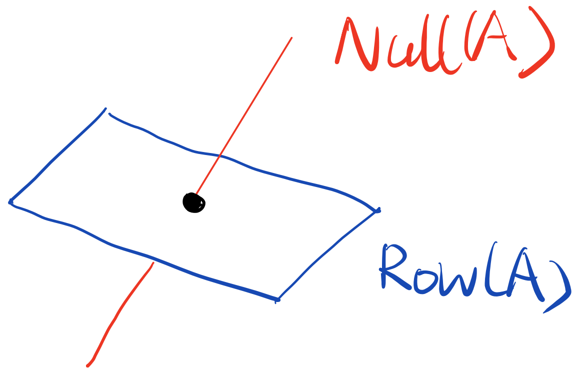
Null(A) is a 1-dimensional subspace of $\mathbb{R}^3$.

Also, $\begin{pmatrix} 1 & 3 & 2 \\ 1 & 2 & 3 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = 0 \quad \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = 0 \quad \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = 0$

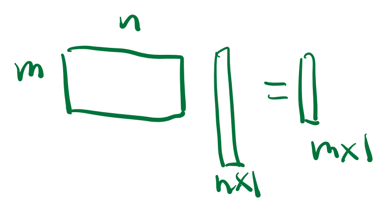
means the dot product of rows of A with $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ are also zero.

If we regard both $\mathbb{R}^3$ and $\mathbb{R}^{1\times 3}$ as the same then the line Null(A) is $90^\circ$ to the plane Row(A).




$\text{Row}(A) \subseteq \mathbb{R}^{1\times 3}$

$\text{Null}(A) \subseteq \mathbb{R}^3$




Example: $A \in \mathbb{R}^{3\times 3}$, all possibilities:

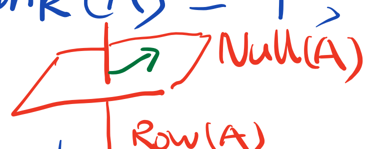
① $\text{rank}(A) = 3$, $\text{nullity}(A) = 0$

$\text{Col}(A) = \mathbb{R}^3$ $\quad$ $\text{Null}(A) = \{\vec{0}\}$

$\text{Row}(A) = \mathbb{R}^{1\times 3}$

② $\text{rank}(A) = 2$, $\text{nullity}(A) = 1$

③ $\text{rank}(A) = 1$, $\text{nullity}(A) = 2$




④ $\text{rank}(A) = 0$, $\text{nullity}(A) = 3$

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The following are equivalent for a square matrix $A \in \mathbb{R}^{n\times n}$ (e.g. $A \in \mathbb{R}^{3\times 3}$):

① $A$ is invertible (nonsingular)

② The homogeneous system $A\vec{x} = \vec{0}$ has only zero sol

$$\begin{pmatrix} 1 & 3 & 2 \\ 1 & 2 & 3 \\ 0 & 1 & -1 \end{pmatrix}$$

③ $\text{Null}(A)$ is trivial

$\text{Null}(A) = \{\vec{0}\}$

$\text{Nullity}(A) = 0$

$$a\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + c\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 & 2 \\ 1 & 2 & 3 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

④ $A\vec{x} = \vec{b}$ has a unique solution.

⑤ Cols of $A$ are independent

Col Rank is $n$

$\text{Rank}(A) = n$

$\text{Col}(A) = \mathbb{R}^n$

⑥ Row of $A$ are independent

Row Rank is $n$

$\text{Rank}(A) = n$ $\quad\quad \text{Row}(A) = \mathbb{R}^{1\times n}$

⑦ $RREF(A) = I$

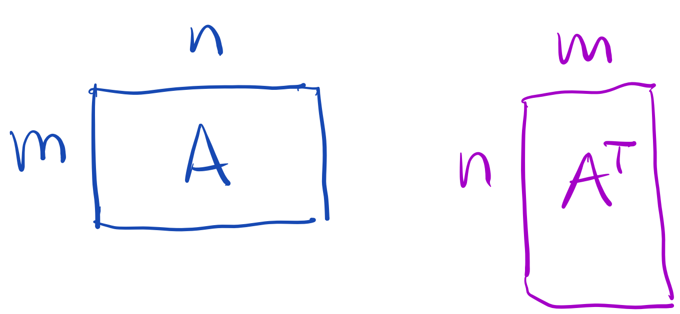
⑧ $\forall \vec{b} \in \mathbb{R}^n, \ \vec{b} \in Col(A) \Rightarrow \mathbb{R}^n \subseteq Col(A) \subseteq \mathbb{R}^n$

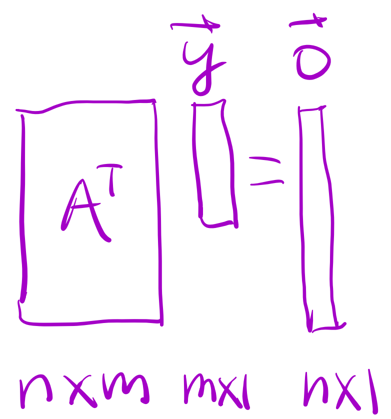
$\Rightarrow Col(A) = \mathbb{R}^n$

**Def** $Null(A^T)$ is also called left null space of $A$

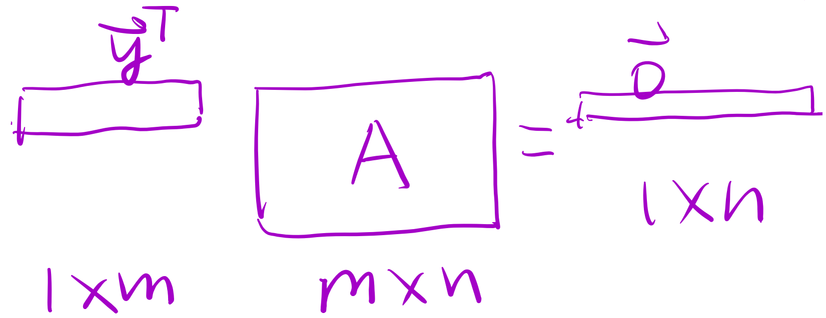
$Null(A^T) = \{\vec{y} \in \mathbb{R}^m : A^T \vec{y} = \vec{0}\}$

$\sim \{\vec{y}^T \in \mathbb{R}^{1 \times m} : \vec{y}^T A = \vec{0}\}$




$(AB)^T = B^T A^T$




Cols of $A^T$ correspond to rows of $A$

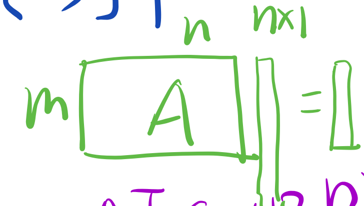
$\Rightarrow Col(A^T)$ correspond to $Row(A)$

Example: $A = \begin{pmatrix} 1 & 3 & 2 \\ 1 & 2 & 3 \\ 0 & 1 & -1 \end{pmatrix}$, $A^T = \begin{pmatrix} 1 & 1 & 0 \\ 3 & 2 & 1 \\ 2 & 3 & -1 \end{pmatrix}$

$\text{Row}(A) = \text{Span}\{[1\ \ 3\ \ 2], [1\ \ 2\ \ 3]\}$

$\text{Col}(A^T) = \text{Span}\left\{\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right\}$

We call both the row space of A.




If $A \in \mathbb{R}^{m\times n}$, $\text{rank}(A) = r \Rightarrow \begin{cases} A^T \in \mathbb{R}^{n\times m} \\ \text{rank}(A^T) = r \end{cases}$

Four subspaces for A.

① $\text{Col}(A)$ is $r$-dim subspace of $\mathbb{R}^m$

$\text{Row}(A^T)$ is $r$-dim subspace of $\mathbb{R}^{1\times m}$

② $\text{Row}(A)$ is $r$-dim subspace of $\mathbb{R}^{1\times n}$

$\text{Col}(A^T)$ is $r$-dim subspace of $\mathbb{R}^n$

③ $\text{Null}(A)$ is $(n-r)$-dim subspace of $\mathbb{R}^n$.

④ $\text{Null}(A^T)$ is $(m-r)$-dim subspace of $\mathbb{R}^m$

$$\underset{m\times n}{A}, \quad \operatorname{rank}(A) = r$$

$$\underset{n\times m}{A^T}\ \underset{m\times 1}{\vec{y}} = \underset{n\times 1}{\vec{0}}$$

$$A^T\vec{y} = \vec{0}$$

$$\operatorname{rank}(A^T) = r$$

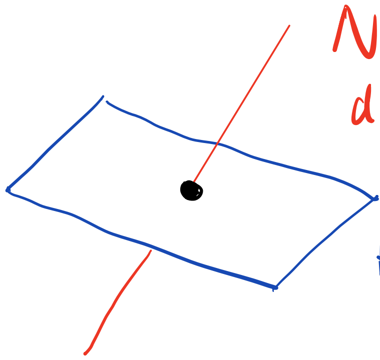
Apply Dimension Theorem to $A^T \in \mathbb{R}^{n\times m}$

$\Rightarrow \operatorname{rank}(A^T) + \operatorname{Nullity}(A^T) = m$

$\Rightarrow \operatorname{Nullity}(A^T) = m - r$

$A \in \mathbb{R}^{m\times n}$

① 



$\operatorname{Null}(A)$

dim $= n - r$

$\operatorname{Row}(A)$

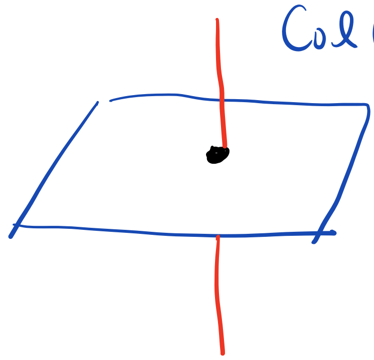
$\subseteq \mathbb{R}^n$

$$\underset{m\times n}{A}\ \underset{n\times 1}{\vec{x}} = \underset{m\times 1}{\vec{0}}$$

---

② 

$\operatorname{Col}(A) = \operatorname{Row}(A^T)$



$\operatorname{Null}(A^T)$

dim $= m - r$

$\subseteq \mathbb{R}^m$

$$\underset{n\times m}{A^T}\ \underset{m\times 1}{\vec{y}} = \underset{n\times 1}{\vec{0}}$$

$$\underset{1\times m}{\vec{y}^T}\ A = \underset{1\times n}{\vec{0}}$$

# Some HW 3&4 Problems

(j) If $A\vec{x} = \vec{b}$ has no solutions at all, then $\vec{b}$ cannot be a linear combination of columns of $A$.
**True.**

(i) For a square matrix $A$, if its rows are linearly independent, then so are its columns.

**True**: if $A$ is $n \times n$, then all rows being linearly independent implies the row rank is $n$, thus the col rank is $n$, which implies all columns are linearly independent.

(j) If $V$ and $W$ are subspaces of $\mathbb{R}^3$, and $dim(V) + dim(W) = 4$, then there must be some nonzero vector in both $V$ and $W$.

**True**.

(g) For $A \in \mathbb{R}^{m \times n}$, the rank of $A^T$ is equal to the rank of $A$.

**True**.

(h) For a square matrix $A$, the nullity of $A^T$ is equal to the nullity of $A$.

**True**: use Dimension Theorem.