Chapter 4 Orthogonality

$\boxed{\text{Def}}$ $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $\vec{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$

The dot product of $\vec{x}$ and $\vec{y}$ is

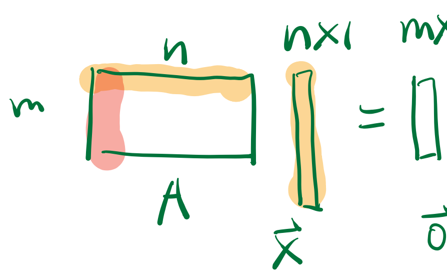
$$\vec{x} \cdot \vec{y} = \langle \vec{x}, \vec{y} \rangle = \vec{y}^T \vec{x}$$

$$= [y_1 \; y_2 \; \cdots \; y_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

$\boxed{\text{Def}}$ We say $\vec{x}$ is orthogonal to $\vec{y}$ if

$$\langle \vec{x}, \vec{y} \rangle = 0.$$



$\mathrm{Row}(A) = \mathrm{Col}(A^T)$

$\boxed{\text{Def}}$ If V, W are two subspaces of the same vector space, we say V is orthogonal to W if any vector in V is orthogonal to any vector in W

Example: $A = \begin{pmatrix} 1 & 3 & 2 \\ 1 & 2 & 3 \\ 0 & 1 & -1 \end{pmatrix}$, $A^T = \begin{pmatrix} 1 & 1 & 0 \\ 3 & 2 & 1 \\ 2 & 3 & -1 \end{pmatrix}$

$\text{Row}(A) = \text{Span}\{[1\ 3\ 2], [1\ 2\ 3]\}$

$\text{Col}(A^T) = \text{Span}\left\{\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right\}$

→ we call them row space of A.

$\text{Col}(A^T) \subseteq \mathbb{R}^3$

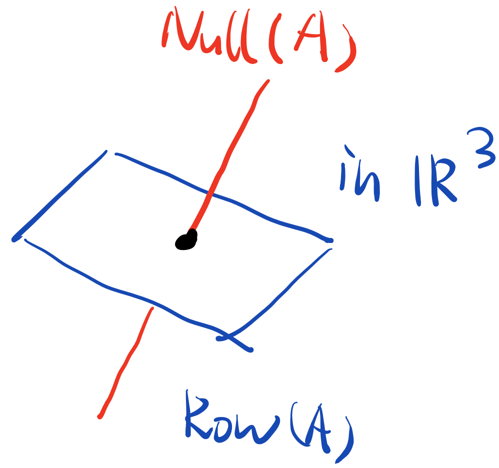
$\text{Null}(A) \subseteq \mathbb{R}^3$

→ they are $\perp$ to each other

$$\begin{pmatrix} 1 & 3 & 2 \\ 1 & 2 & 3 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Leftrightarrow \begin{cases} \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \\ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \\ \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \end{cases}$$

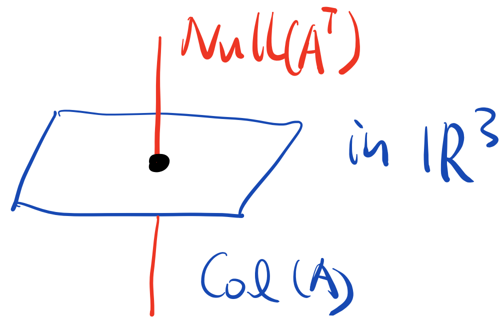
$$A = \begin{pmatrix} 1 & 3 & 2 \\ 1 & 2 & 3 \\ 0 & 1 & -1 \end{pmatrix}$$



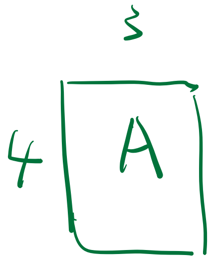
$$A^T = \begin{pmatrix} 1 & 1 & 0 \\ 3 & 2 & 1 \\ 2 & 3 & -1 \end{pmatrix}$$

$\mathrm{Null}(A^T)$

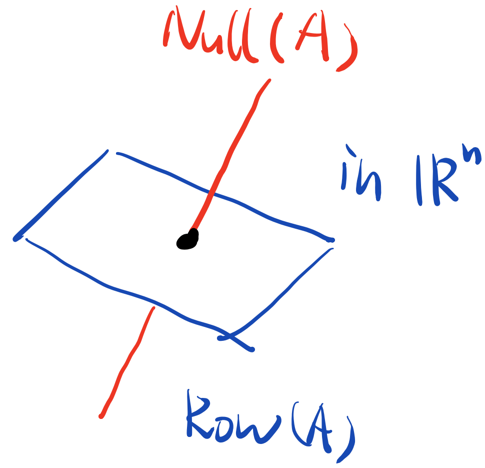
$= \{ \vec{y} \in \mathbb{R}^3 : A^T \vec{y} = \vec{0} \}$

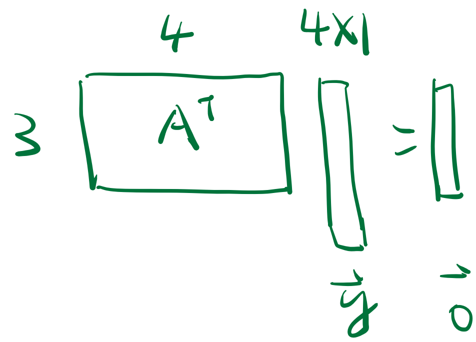


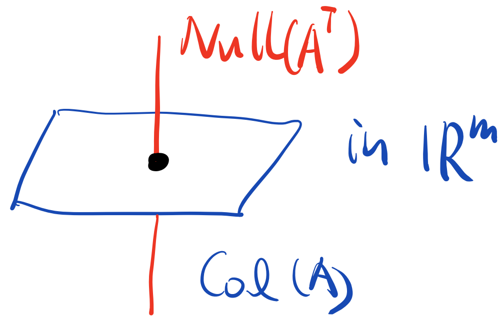
$\mathrm{Rank}(A) = 2 \Rightarrow \mathrm{Rank}(A^T) = 2$



$\mathrm{Col}(A) = \mathrm{Row}(A^T)$







Dimension Thm for $A^T\vec{y} = \vec{0}$

$\Rightarrow \dim(\mathrm{Null}(A^T)) = 4 - 2 = 2$

For a matrix $A \in \mathbb{R}^{m \times n}$, if $\mathrm{rank}(A) = r$,

$$\dim(\mathrm{Null}(A)) = n - r$$

$$\dim(\mathrm{Null}(A^T)) = m - r$$

- Dot (Inner) Product: $\vec{x}, \vec{y} \in \mathbb{R}^n$

$$\langle \vec{x}, \vec{y} \rangle = \vec{y}^T \vec{x} = (y_1 \; y_2 \cdots y_n) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$= \sum_{i=1}^{n} x_i y_i$$

$$= x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

- Length: $\|\vec{x}\| = \sqrt{\langle \vec{x}, \vec{x} \rangle} = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$

- Orthogonality: we say $\vec{x} \perp \vec{y}$ if $\langle \vec{x}, \vec{y} \rangle = 0$

- Properties of Dot Product: $\forall \vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^n$, $a, b \in \mathbb{R}$

① Linearity: $\langle a\vec{x} + b\vec{y}, \vec{z} \rangle = a\langle \vec{x}, \vec{z} \rangle + b\langle \vec{y}, \vec{z} \rangle$

$\langle \vec{z}, a\vec{x} + b\vec{y} \rangle = a\langle \vec{z}, \vec{x} \rangle + b\langle \vec{z}, \vec{y} \rangle$

② Symmetry: $\langle \vec{x}, \vec{y} \rangle = \langle \vec{y}, \vec{x} \rangle$

③ $\langle \vec{x}, \vec{x} \rangle \geq 0$

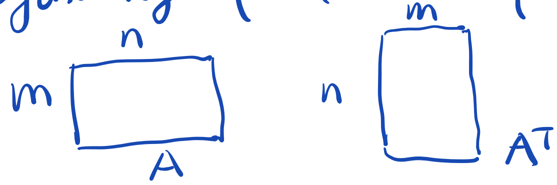
and

$\langle \vec{x}, \vec{x} \rangle = 0 \iff \vec{x} = \vec{0}$

Any operation on two vectors satisfying these 3 properties is called an Inner Product.

Dot product is the simplest inner product.

Orthogonality of four subspaces of $A \in \mathbb{R}^{m \times n}$:



① Column Space of A : $Col(A) \subseteq \mathbb{R}^m$

② Row Space of A : $Col(A^T) \subseteq \mathbb{R}^n$

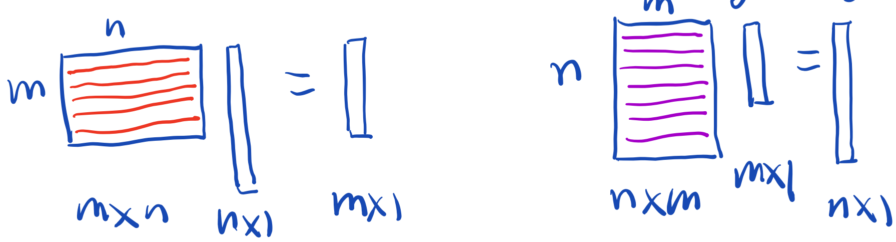
③ Null Space of A : $Null(A) \subseteq \mathbb{R}^n$

④ Left Null Space of A : $Null(A^T) \subseteq \mathbb{R}^m$

$$Null(A) = \{\vec{x} \in \mathbb{R}^n : \boxed{A\vec{x} = \vec{0}}\}$$

$$Null(A^T) = \{\vec{y} \in \mathbb{R}^m : A^T\vec{y} = \vec{0}\}$$

$A\vec{x} = \vec{0}$ $\qquad\qquad$ $A^T\vec{y} = \vec{0}$



If $A\vec{x} = \vec{0}$, then each row of A is $\perp$ to $\vec{x}$

$\Rightarrow \boxed{Col(A^T)} \perp Null(A)$

If $A^T\vec{y} = \vec{0}$, then each row of A^T is $\perp$ to $\vec{y}$

$\Rightarrow$ each col of A is $\perp$ to $\vec{y}$

$\Rightarrow Col(A) \perp Null(A^T)$

Example: Find four subspaces of

$$A = \begin{pmatrix} 1 & 3 & 2 & 2 \\ 1 & 2 & 3 & 1 \\ 0 & 1 & -1 & 1 \end{pmatrix}, \quad A^T = \begin{pmatrix} 1 & 1 & 0 \\ 3 & 2 & 1 \\ 2 & 3 & -1 \\ 2 & 1 & 1 \end{pmatrix}$$

3×4

$$\mathrm{RREF}(A|\vec{0}) \quad \left(\begin{array}{cccc|c} 1 & 3 & 2 & 2 & 0 \\ 1 & 2 & 3 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{cccc|c} 1 & 3 & 2 & 2 & 0 \\ 0 & -1 & 1 & -1 & 0 \\ 0 & 1 & -1 & 1 & 0 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{cccc|c} 1 & 3 & 2 & 2 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\rightarrow \begin{array}{c} \left(\begin{array}{cccc|c} 1 & 0 & 5 & -1 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \\ \begin{array}{cccc} x & y & z & w \end{array} \end{array}$$

$$\begin{cases} z = s \\ w = t \end{cases} \Rightarrow \begin{cases} x = -5s + t \\ y = s - t \end{cases} \Rightarrow \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -5s+t \\ s-t \\ s \\ t \end{pmatrix}$$

$$= \begin{pmatrix} -5s \\ s \\ s \\ 0 \end{pmatrix} + \begin{pmatrix} t \\ -t \\ 0 \\ t \end{pmatrix} = s\begin{pmatrix} -5 \\ 1 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \quad \forall s,t \in \mathbb{R}$$

$$\Rightarrow \mathrm{Null}(A) = \mathrm{Span}\left\{\begin{bmatrix} -5 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}\right\} \subseteq \mathbb{R}^4$$

$$\mathrm{Col}(A) = \mathrm{Span}\left\{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}\right\} \subseteq \mathbb{R}^3$$

Finding Row Space Basis :

$$\begin{array}{r}
a \,[\,1 \quad 3 \quad 2 \quad 2\,] \\
+\,b \,[\,1 \quad 2 \quad 3 \quad 1\,] \\
+\,c \,[\,0 \quad 1 \quad -1 \quad 1\,]
\end{array} = [\,0 \quad 0 \quad 0 \quad 0\,]$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 3 & 2 & 1 \\ 2 & 3 & -1 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\, A^T \;\middle|\; \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \end{matrix} \,\right]$$

$\mathrm{RREF}(A^T \mid \vec{0})$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 2 & 3 & -1 & 0 \\ 2 & 1 & 1 & 0 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\longrightarrow \begin{array}{c} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \\ \begin{array}{ccc} x & y & z \end{array} \end{array}$$

$$z = t \Rightarrow \begin{cases} x = -t \\ y = t \end{cases} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -t \\ t \\ t \end{pmatrix} = t\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \quad \forall t \in \mathbb{R}$$

$$\Rightarrow \mathrm{Null}(A^T) = \mathrm{Span}\left\{\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}\right\} \subseteq \mathbb{R}^3$$

$$\mathrm{Col}(A^T) = \mathrm{Span}\left\{\begin{pmatrix} 1 \\ 3 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ 1 \end{pmatrix}\right\} \subseteq \mathbb{R}^4$$

$\mathrm{rank}(A) = \mathrm{rank}(A^T) = 2$

$\mathrm{Col}(A^T) \perp \mathrm{Null}(A)$

$$\mathrm{Null}(A) = \mathrm{Span}\left\{ \begin{bmatrix} -5 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\} \subseteq \mathbb{R}^4$$

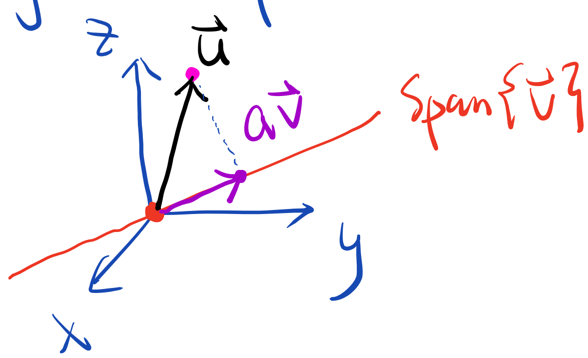
$A: 3\times 4$

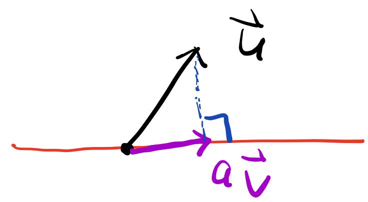
$A^T: 4\times 3$

$$\mathrm{Col}(A) = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \right\} \subseteq \mathbb{R}^3$$

$\mathrm{Col}(A) \perp \mathrm{Null}(A^T)$

Projection of a vector onto another one:





So we look for a vector $a\vec{v}$ satisfying

$$(\vec{u} - a\vec{v}) \perp \vec{v}$$

$$\Leftrightarrow \quad \langle \vec{u} - a\vec{v}, \vec{v} \rangle = 0$$

$$\Leftrightarrow \quad \langle \vec{u}, \vec{v} \rangle + \langle -a\vec{v}, \vec{v} \rangle = 0$$

$$\Leftrightarrow \quad \langle \vec{u}, \vec{v} \rangle - a\langle \vec{v}, \vec{v} \rangle = 0$$

$$\Leftrightarrow \quad a\langle \vec{v}, \vec{v} \rangle = \langle \vec{u}, \vec{v} \rangle$$

$$\Leftrightarrow \quad a = \frac{\langle \vec{u}, \vec{v} \rangle}{\langle \vec{v}, \vec{v} \rangle} = \frac{\vec{v}^T \vec{u}}{\vec{v}^T \vec{v}} = \frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}}$$

$$= \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2}$$

For $\vec{u}, \vec{v} \in \mathbb{R}^n$, the projection of $\vec{u}$ onto $\vec{v}$

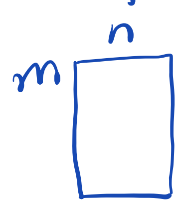
is $P_{\vec{v}}(\vec{u}) = \dfrac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\vec{v}$

Ex: Projection of $\vec{u} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ onto $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$P_{\vec{v}}(\vec{u}) = \frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\vec{v} = \frac{6}{(1+1+1)}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

Motivation for Projection

$A\vec{x} = \vec{b}$ $\qquad A \in \mathbb{R}^{m\times n}$



$m > n$

$A\vec{x} = \vec{b}$ has at least one sol

$\Longleftrightarrow \vec{b} \in Col(A) \subseteq \mathbb{R}^m$

$\dim(Col(A)) \leq n < m$

$\vec{b} \in \mathbb{R}^m$, $\quad Col(A) \neq \mathbb{R}^m$