- A set of column vectors $\{\vec{a}_1, \dots, \vec{a}_n\}$ is called orthogonal if $\vec{a}_i \perp \vec{a}_j$ for any $i, j$.

$$\langle \vec{a}_i, \vec{a}_j \rangle = 0$$
$$\vec{a}_j^T \vec{a}_i = 0$$

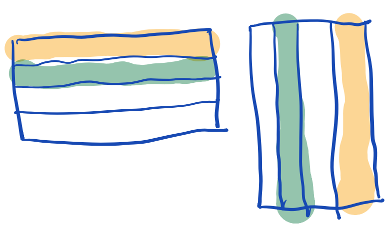
Ex: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$, $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\}$

- A vector $\vec{u}$ is unit if $\|\vec{u}\| = 1$.

- A set of column vectors $\{\vec{a}_1, \dots, \vec{a}_n\}$ is called orthonormal if $\begin{cases} \vec{a}_i \perp \vec{a}_j \text{ for any } i, j. \\ \|\vec{a}_i\| = 1, \ \forall i \end{cases}$

$$\vec{a}_j^T \vec{a}_i = \begin{cases} 0, & i \neq j \\ 1, & i = j. \end{cases}$$

- Let $\vec{a}_1, \dots, \vec{a}_n$ be cols of $A \in \mathbb{R}^{m \times n}$ assume they are orthonormal, then

$A^T A =$  $= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

In this case, the projection matrix is

$$A(A^TA)^{-1}A^T = AA^T.$$

- Let $\{\vec{a}_1, \cdots, \vec{a}_n\} \subseteq \mathbb{R}^m$, want to generate $\{\vec{c}_1, \cdots, \vec{c}_n\}$ which has the same span and is orthornormal.

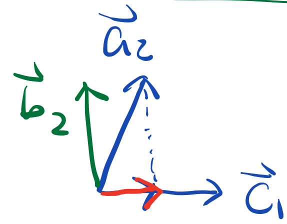
Gram-Schmidt Process

**Line Projection**

$$P_{\vec{v}}(\vec{u}) = \frac{\langle \vec{u}, \vec{v} \rangle}{\langle \vec{v}, \vec{v} \rangle} \vec{v}$$

① $\vec{b}_1 = \vec{a}_1$

$\vec{c}_1 = \dfrac{\vec{b}_1}{\|\vec{b}_1\|}$

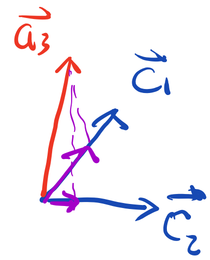



② $\vec{b}_2 = \vec{a}_2 - \langle \vec{a}_2, \vec{c}_1 \rangle \vec{c}_1$

$\vec{c}_2 = \dfrac{\vec{b}_2}{\|\vec{b}_2\|}$

③ $\vec{b}_3 = \vec{a}_3 - \langle \vec{a}_3, \vec{c}_1 \rangle \vec{c}_1 - \langle \vec{a}_3, \vec{c}_2 \rangle \vec{c}_2$

$\vec{c}_3 = \dfrac{\vec{b}_3}{\|\vec{b}_3\|}$



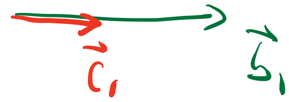

④ $\vec{b}_4 = \vec{a}_4 - \langle \vec{a}_4, \vec{c}_1 \rangle \vec{c}_1 - \langle \vec{a}_4, \vec{c}_2 \rangle \vec{c}_2 - \langle \vec{a}_4, \vec{c}_3 \rangle \vec{c}_3$

$\vec{c}_4 = \dfrac{\vec{b}_4}{\|\vec{b}_4\|}$

Ex: Find orthonormal basis for $\mathrm{Span}\left\{ \underset{\vec{a}_1}{\begin{bmatrix}1\\1\\1\end{bmatrix}}, \underset{\vec{a}_2}{\begin{bmatrix}3\\2\\1\end{bmatrix}} \right\}$

Sol: ① $\vec{b}_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}$

$$\vec{c}_1 = \frac{\vec{b}_1}{\|\vec{b}_1\|} = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}$$




② $\vec{b}_2 = \vec{a}_2 - \langle \vec{a}_2, \vec{c}_1 \rangle \vec{c}_1$

$$\vec{b}_2 = \vec{a}_2 - \frac{\langle \vec{a}_2, \vec{b}_1 \rangle}{\langle \vec{b}_1, \vec{b}_1 \rangle}\vec{b}_1$$

$$= \begin{bmatrix}3\\2\\1\end{bmatrix} - \left\langle \begin{bmatrix}3\\2\\1\end{bmatrix}, \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix} \right\rangle \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}$$

$$= \begin{bmatrix}3\\2\\1\end{bmatrix} - \frac{6}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}$$

$$= \begin{bmatrix}3\\2\\1\end{bmatrix} - \begin{bmatrix}2\\2\\2\end{bmatrix} = \begin{bmatrix}1\\0\\-1\end{bmatrix}$$

$$\vec{c}_2 = \frac{\vec{b}_2}{\|\vec{b}_2\|} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\-1\end{bmatrix}$$