Plan: ① Diagonalization
② Matrix Exponential
③ Linear ODE

- Consider $A \in \mathbb{R}^{6\times6}$, assume

$$\det(A-\lambda I) = (\lambda-a)^2(\lambda-b)(\lambda-c)^3$$

definition of alg mul
$\lambda = a$ has algebraic multiplicity 2
$\lambda = b$ has algebraic multiplicity 1
$\lambda = c$ has algebraic multiplicity 3

The dimension of each eigen-space is called geometric multiplicity of each eigenvalue.

$\boxed{\text{Theorem}}$ $1 \le$ Geometric Mul $\le$ Algebraic Mul

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Diagonalization of Matrices

For $A \in \mathbb{R}^{n\times n}$, assume it has n linearly independent eigenvectors (not always true)

$\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$ with eigenvalues

$\lambda_1, \lambda_2, \cdots, \lambda_n$ (could be repeated ones),

then $A\vec{v}_i = \lambda_i \vec{v}_i$ can imply diagonalization as follows:

$$A = \begin{pmatrix} 5 & 2 & 0 \\ 2 & 5 & 0 \\ -3 & 4 & 6 \end{pmatrix}, \quad \lambda_1 = 6, \; \lambda_2 = 7, \; \lambda_3 = 3$$

$$\vec{v}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \; \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \; \vec{v}_3 = \begin{pmatrix} 3 \\ -3 \\ 7 \end{pmatrix}$$

$$A\vec{v}_i = \lambda_i \vec{v}_i, \quad i = 1, 2, 3$$

$$[A\vec{v}_1 \;\; A\vec{v}_2 \;\; A\vec{v}_3] = [\lambda_1\vec{v}_1 \;\; \lambda_2\vec{v}_2 \;\; \lambda_3\vec{v}_3]$$

$$A[\vec{v}_1 \;\; \vec{v}_2 \;\; \vec{v}_3] = [\vec{v}_1 \;\; \vec{v}_2 \;\; \vec{v}_3]\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$$\begin{pmatrix} 5 & 2 & 0 \\ 2 & 5 & 0 \\ -3 & 4 & 6 \end{pmatrix}\begin{pmatrix} 0 & 1 & 3 \\ 0 & 1 & -3 \\ 1 & 1 & 7 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 3 \\ 0 & 1 & -3 \\ 1 & 1 & 7 \end{pmatrix}\begin{pmatrix} 6 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$A\vec{v}_1 = \lambda_1\vec{v}_1$$

$$A \; V = V \cdot D$$

$AV = VD \Rightarrow \underline{A = VDV^{-1}}$ diagonalization

Independent eigenvectors $\Rightarrow$ V has independent cols

$\Rightarrow$ col(V) is n-dimensional

$\Rightarrow$ rank$(V) = n$

$\Rightarrow$ V^{-1} exists

Find V^{-1} by Gaussian Elimination or Cofactor Matrix:

$$V^{-1} = \begin{pmatrix} -2/3 & -1/3 & 1 \\ 1/2 & 1/2 & 0 \\ 1/6 & -1/6 & 0 \end{pmatrix}$$

$A = VDV^{-1}$ is called diagonalization of A.

There are many applications of diagonalization.

Example: $A^2 = A \cdot A = VDV^{-1}VDV^{-1}$

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

$$= V D \cdot D V^{-1}$$

$$= V D^2 V^{-1}$$

$$D^2 = \begin{pmatrix} \lambda_1^2 & & \\ & \lambda_2^2 & \\ & & \lambda_3^2 \end{pmatrix}$$

$$= V \begin{pmatrix} \lambda_1^2 & & \\ & \lambda_2^2 & \\ & & \lambda_3^2 \end{pmatrix} V^{-1}$$

$$A^n = V \begin{pmatrix} \lambda_1^n & & \\ & \lambda_2^n & \\ & & \lambda_3^n \end{pmatrix} V^{-1}$$

We can also define e^A by

$$e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots$$

$$= VIV^{-1} + VDV^{-1} + \frac{1}{2!}VD^2V^{-1} + \frac{1}{3!}VD^3V^{-1} + \cdots$$

$$= V\left[I + D + \frac{1}{2!}D^2 + \frac{1}{3!}D^3 + \cdots\right]V^{-1}$$

$$= V \begin{pmatrix} 1+\lambda_1+\frac{1}{2!}\lambda_1^2+\frac{1}{3!}\lambda_1^3+\cdots & 0 & 0 \\ 0 & 1+\lambda_2+\frac{1}{2!}\lambda_2^2+\frac{1}{3!}\lambda_2^3+\cdots & 0 \\ 0 & 0 & 1+\lambda_3+\frac{1}{2!}\lambda_3^2+\frac{1}{3!}\lambda_3^3+\cdots \end{pmatrix} V^{-1}$$

$$= V \begin{pmatrix} e^{\lambda_1} & 0 & 0 \\ 0 & e^{\lambda_2} & 0 \\ 0 & 0 & e^{\lambda_3} \end{pmatrix} V^{-1}$$

$$\underset{A}{\begin{pmatrix} 5 & 2 & 0 \\ 2 & 5 & 0 \\ -3 & 4 & 6 \end{pmatrix}} \underset{V}{\begin{pmatrix} 0 & 1 & 3 \\ 0 & 1 & -3 \\ 1 & 1 & 7 \end{pmatrix}} = \underset{V}{\begin{pmatrix} 0 & 1 & 3 \\ 0 & 1 & -3 \\ 1 & 1 & 7 \end{pmatrix}} \underset{D}{\begin{pmatrix} 6 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 3 \end{pmatrix}}$$

$$\Rightarrow \quad e^{A} = \begin{pmatrix} 0 & 1 & 3 \\ 0 & 1 & -3 \\ 1 & 1 & 7 \end{pmatrix} \begin{pmatrix} e^6 & 0 & 0 \\ 0 & e^7 & 0 \\ 0 & 0 & e^3 \end{pmatrix} \begin{pmatrix} 0 & 1 & 3 \\ 0 & 1 & -3 \\ 1 & 1 & 7 \end{pmatrix}^{-1}.$$

$A = VDV^{-1} \Rightarrow$

Diagonalization & Matrix Exponential can be used for solving linear differential equations such as

2nd order equation
$$\begin{cases} 2y''(t) + 3y'(t) + y(t) = 0 \\ y(0) = 1, \quad y'(0) = 2 \end{cases}$$

① 1st order equation

$$\begin{cases} y'(t) - a\,y(t) = 0 \\ y(0) = b \end{cases}$$

$$y'(t) = a\,y(t)$$

$$\frac{y'(t)}{y(t)} = a$$

$$\int \frac{y'(t)}{y(t)}\,dt = \int a\,dt$$

$$\ln[y(t)] = at + C$$

$$y(t) = e^{at+C}$$

$$y(0) = b \Rightarrow e^{C} = b \Rightarrow y(t) = e^{at} b$$

$$y(t) = e^{at} y(0)$$

②

$$2y''(t) + 3y'(t) + y(t) = 0$$

$$\vec{u}(t) = \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix}, \quad \frac{d}{dt}\vec{u}(t) = \begin{pmatrix} y'(t) \\ y''(t) \end{pmatrix}$$

$$\Rightarrow \vec{u}'(t) = \begin{pmatrix} y'(t) \\ -\frac{3}{2}y'(t) - \frac{1}{2}y(t) \end{pmatrix}$$

$$\Rightarrow \frac{d}{dt}\begin{pmatrix} y(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{1}{2} & -\frac{3}{2} \end{pmatrix}\begin{pmatrix} y(t) \\ y'(t) \end{pmatrix}$$ first order system

$$\frac{d}{dt}\vec{u}(t) = A\,\vec{u}(t) \qquad \begin{cases} y'(t) = a\,y(t) \\ y(t) = e^{at}\,y(0) \end{cases}$$

$$\vec{u}(t) = e^{At}\,\vec{u}(0)$$

$$\left(\frac{d}{dt}y(t) = a\,y(t) \Rightarrow y(t) = e^{at}\,y(0) \right)$$

To see why $\vec{u}(t) = e^{At}\,\vec{u}(0)$

for a diagonalizable matrix $A = VDV^{-1}$:

$$A = \begin{pmatrix} 0 & 1 \\ -\frac{1}{2} & -\frac{3}{2} \end{pmatrix}$$

1) Compute Eigenvalues

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ -\frac{1}{2} & -\frac{3}{2}-\lambda \end{vmatrix} = \lambda\left(\lambda + \frac{3}{2}\right) + \frac{1}{2}$$

$$= \lambda^2 + \frac{3}{2}\lambda + \frac{1}{2}$$

$$= \frac{1}{2}\left[2\lambda^2 + 3\lambda + 1\right]$$

$$= \frac{1}{2}(2\lambda+1)(\lambda+1)$$

$$\Rightarrow \lambda_1 = -\frac{1}{2}, \quad \lambda_2 = -1$$

2) Find Eigen-Spaces

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$\lambda_1 = -\frac{1}{2}, \qquad (A - \lambda I)\vec{v} = \vec{0}$$

$$\left[\begin{array}{cc|c} \frac{1}{2} & 1 & 0 \\ -\frac{1}{2} & -1 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$v_2 = t \Rightarrow v_1 = -2t$$

$$\Rightarrow \vec{v} = \begin{pmatrix} -2t \\ t \end{pmatrix} = t\begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$\Rightarrow$ Eigen-Space for $\lambda_1 = -\frac{1}{2}$ is $\text{Span}\left\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right\}$

$\lambda_2 = -1 \quad (A - \lambda I)\vec{v} = \vec{0} \qquad \vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$v_2 = t \Rightarrow v_1 = -t \Rightarrow \vec{v} = \begin{pmatrix} -t \\ t \end{pmatrix} = t\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$\Rightarrow$ Eigen-Space for $\lambda_2 = -1$ is $\text{Span}\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$

3) So A is diagonalizable

$$A = VDV^{-1} = \begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -1 & -1 \\ +1 & 2 \end{bmatrix}$$

$$At = tVDV^{-1} = V(tD)V^{-1}$$

$$= \begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{2}t & 0 \\ 0 & -t \end{bmatrix} \begin{bmatrix} -1 & -1 \\ +1 & 2 \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{-\frac{1}{2}t} & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix}$$

$$\vec{u}'(t) = A\vec{u}(t) \qquad \frac{d}{dt}\begin{pmatrix} y(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{1}{2} & -\frac{3}{2} \end{pmatrix}\begin{pmatrix} y(t) \\ y'(t) \end{pmatrix}$$

$$A = VDV^{-1}$$

$$A = VDV^{-1} = \begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ +1 & 2 \end{bmatrix}$$

$$\vec{u}'(t) = VDV^{-1}\vec{u}(t)$$

$$V^{-1}\vec{u}'(t) = DV^{-1}\vec{u}(t)$$

$$\frac{d}{dt}\Big[\underbrace{V^{-1}\vec{u}(t)}_{\vec{w}(t)}\Big] = D\left[V^{-1}\vec{u}(t)\right]$$

$$\frac{d}{dt}\vec{w}(t) = D\vec{w}(t)$$

$$\begin{pmatrix} w_1'(t) \\ w_2'(t) \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\begin{pmatrix} w_1(t) \\ w_2(t) \end{pmatrix}$$

$$\Rightarrow \begin{cases} w_1'(t) = \lambda_1 w_1(t) \Rightarrow w_1(t) = e^{\lambda_1 t} w_1(0) \\ w_2'(t) = \lambda_2 w_2(t) \Rightarrow w_2(t) = e^{\lambda_2 t} w_2(0) \end{cases}$$

$$\Rightarrow \begin{pmatrix} w_1(t) \\ w_2(t) \end{pmatrix} = \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} \begin{pmatrix} w_1(0) \\ w_2(0) \end{pmatrix}$$

$\vec{w} = V^{-1} \vec{u}$

$\vec{u} = V \vec{w}$

$$\Rightarrow \vec{u}(t) = V \vec{w}(t) = V \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} \vec{w}(0)$$

$$= \underbrace{V \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} V^{-1}} \vec{u}(0)$$

$$= e^{At} \vec{u}(0)$$