Today's Plan

① Review

② Diagonalization

③ Differential Equations

Review :

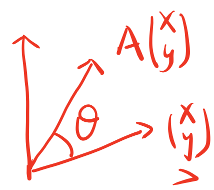
- $|A-\lambda I| = \det(A-\lambda I)$ is characteristic polynomial of A.

  The roots of this polynomial are eigenvalues of A.

- For $A \in \mathbb{R}^{n\times n}$, Foundamental Theorem of Algebra implies

  $\det(A-\lambda I)$ has n complex roots (including reapted ones).

  It may not have real roots.



Example: $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

$$|A-\lambda I| = \lambda^2 - 2\cos\theta\,\lambda + 1$$

Discriminant of a quadratic poly is

$$\Delta = 4\cos^2\theta - 4 = -4\sin^2\theta$$

So $\begin{cases} \Delta < 0 & \text{for } \theta \neq 0, \pi \Rightarrow \text{no rea roots} \\ \Delta = 0 & \text{for } \begin{cases} \theta = 0 \Rightarrow \lambda = 1 \\ \theta = \pi \Rightarrow \lambda = -1 \end{cases} \end{cases}$

So no real eigenvalues (thus no real eigenvectors)

for A if $\theta \neq 0, \pi$.

But A always has complex eigenvalues for

any θ.

- If an eigenvalue λ is a root reapted m times for

the characteristic polynomial $p(t) = \det(A - tI)$,
then m is called algebraic multiplicity of λ

Example: $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$, $\det(A - \lambda I) = (2-\lambda)^3$

1) The polynomial has 3 reapted roots

2) There is only one eigenvalue $\lambda = 2$ with algebraic multiplity 3.

- Dimension of each eigen-space of an eigenvalue λ is called geometric multiplicity of λ

- Theorem: for each eigenvalue, $\underline{1 \leq}$ Geo Mul $\leq$ Alg Mul.

Diagonalization

- Diagonalization of $A \in \mathbb{R}^{n \times n}$ means that we can find an invertible matrix $V \in \mathbb{R}^{n \times n}$ or $\mathbb{C}^{n \times n}$ and a digonal matrix $D \in \mathbb{R}^{n \times n}$ or $\mathbb{C}^{n \times n}$ s.t.

$$A = V D V^{-1}.$$

- Not all matrices can be diagonalized.

- Assume $A = V D V^{-1}$ is true, want to see what it is equivalent to.

Let cols of V be $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$

Let diagonal entries of D be $d_1, \cdots, d_n$

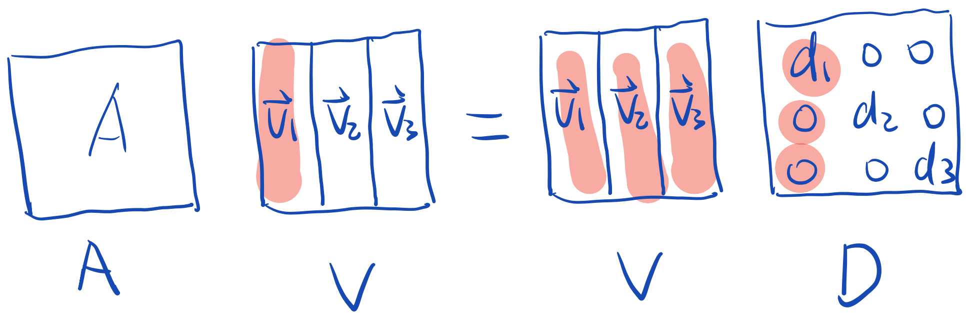
Take $n=3$ as an example:

$$V = [\vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3]$$

$$D = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$$

$$A = V D V^{-1}$$

$$\Leftrightarrow \quad AV = VD$$

$\Leftrightarrow$ 

$$\Leftrightarrow \begin{cases} A\vec{v}_1 = d_1\vec{v}_1 \\ A\vec{v}_2 = d_2\vec{v}_2 \\ A\vec{v}_3 = d_3\vec{v}_3 \end{cases}$$

Since $\vec{v}_i \neq \vec{0}$ (why?), $\vec{v}_i$ is an eigenvector.

$\boxed{\text{Theorem}}$ Let cols of V be $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$

Let diagonal entries of D be $d_1, \cdots, d_n$

Then $A = VDV^{-1}$ if and only if

$\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$ are linearly independent eigenvectors of A, with eigenvalues $d_1, \cdots, d_n$.

$\boxed{\text{Theorem}}$ If $\vec{v}_1, \cdots, \vec{v}_m$ are eigenvectors of A with distinct eigenvalues $\lambda_1, \cdots, \lambda_m$, then they are linearly independent

Remark: If we have n distinct eigenvalues for $A \in \mathbb{R}^{n \times n}$, then we only need to find one eigenvector $\vec{v}_i$ for λ_i.

$$V = [\vec{v}_1\ \vec{v}_2 \cdots \vec{v}_n], \quad D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

$$\Rightarrow A = VDV^{-1}.$$

Example:

$$A = \begin{pmatrix} 5 & 2 & 0 \\ 2 & 5 & 0 \\ -3 & 4 & 6 \end{pmatrix}, \quad \lambda_1 = 6, \quad \lambda_2 = 7, \quad \lambda_3 = 3$$

$$\vec{v}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} 3 \\ -3 \\ 7 \end{pmatrix}$$

$$V = \begin{pmatrix} 0 & 1 & 3 \\ 0 & 1 & -3 \\ 1 & 1 & 7 \end{pmatrix}, \quad D = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$\Rightarrow \quad A = VDV^{-1}$$

$$= \begin{pmatrix} 0 & 1 & 3 \\ 0 & 1 & -3 \\ 1 & 1 & 7 \end{pmatrix} \begin{pmatrix} 6 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 0 & 1 & 3 \\ 0 & 1 & -3 \\ 1 & 1 & 7 \end{pmatrix}^{-1}.$$

- If we don't have n distinct eigenvalues, then we simply collect basis vectors for all eigenspaces:

  ① If we can have n vectors, then diagonalizable

  ② If not, then A is not diagonalizable

Remark: If we allow complex eigenvalues, then we have n eigenvalues including repeated ones. This means

$A \in \mathbb{R}^{6\times 6}$

$(\lambda-a)^3(\lambda-b)^2(\lambda-c)$

$$\sum_i (\text{Alg Mul of } \lambda_i) = n$$

$$\Rightarrow \underbrace{\sum_i (\text{Geo Mul of } \lambda_i)}_{} \le n$$

number of basis vectors for all eigenspaces.

So diagonalization $\Leftrightarrow$ Geo Mul = Alg Mul for all eigenvalues.

Example: $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

$$|A-\lambda I| = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda)^2$$

$$\Rightarrow \begin{cases} \lambda=1 \text{ has alg mul } 1 \\ \lambda=2 \text{ has alg mul } 2 \end{cases}$$

Eigenspace of $\lambda=1$ is $\text{Span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}\right\}$

Eigenspace of $\lambda=2$ is $\text{Span}\left\{\begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$

So A is not diagonalizable.

Example: $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1$$

No real roots $\Rightarrow$ no real eigenvector

$\Rightarrow$ real diagonalization is not possible

$\lambda = \pm i \Rightarrow$ two distinct complex eigenvalues

$i^2 = -1$

$\Rightarrow$ complex diagonalization exists.

① Plug $\lambda = i$ into $(A - \lambda I)\vec{v} = \vec{0}$

$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ $\quad \begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\left[\begin{array}{cc|c} -i & -1 & 0 \\ 1 & -i & 0 \end{array}\right]$$

$i = \sqrt{-1}$

$$\left[\begin{array}{cc|c} 1 & -i & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow$ Eigenspace is $\text{Span}\left\{\begin{bmatrix} i \\ 1 \end{bmatrix}\right\} = \left\{ t\begin{bmatrix} i \\ 1 \end{bmatrix}, \forall t \in \mathbb{C} \right\}$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ i \end{bmatrix} = i\begin{bmatrix} i \\ 1 \end{bmatrix}$$

② $\lambda = -i$ has eigenspace $\text{Span}\left\{\begin{bmatrix} -i \\ 1 \end{bmatrix}\right\}$

So $A = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}\begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}^{-1}$

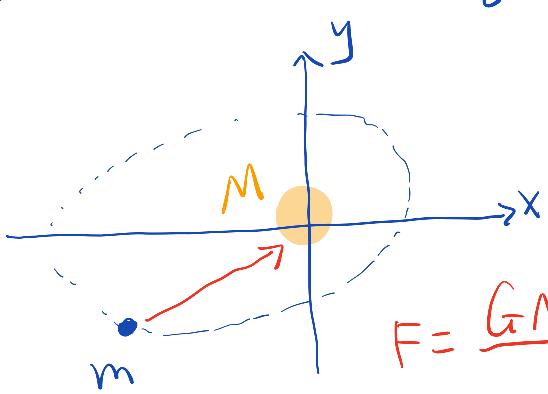
- Ordinary Differential Equations (ODE)

Equations like $y''(t) = -y'(t) + y(t)$ is called ODE

The variable t usually means time in many applications.

Example: Planet Motion by Newton's Law

$F = ma$



$F = \dfrac{GMm}{r^2}$

In a 2D Universe, the position of planet is $\vec{x}(t) = (x(t), y(t))$

$\vec{v}(t) = \vec{x}\,'(t) = (x'(t), y'(t))$ is the velocity

$\vec{a}(t) = \vec{v}\,'(t) = \vec{x}\,''(t) = (x''(t), y''(t))$ is the acceleration.

$$F = \frac{GMm}{r^2} = \frac{GMm}{\|\vec{x}\|^2} = \frac{GMm}{[x(t)]^2 + [y(t)]^2}$$

Directional Force:

$$\vec{F} = F \cdot \left(-\frac{\vec{x}}{\|\vec{x}\|}\right) = \frac{-GMm}{\|\vec{x}\|^3}\vec{x}$$

$$\vec{F} = m\vec{a} \Rightarrow -\frac{GMm}{\|\vec{x}\|^3}\vec{x} = m(x''(t), y''(t))$$

(FYI only; NOT Required)

$$\Rightarrow \begin{cases} x''(t) = \dfrac{-GM}{[x(t)^2 + y(t)^2]^{3/2}} x(t) \\[2ex] y''(t) = \dfrac{-GM}{[x(t)^2 + y(t)^2]^{3/2}} y(t) \end{cases}$$

This is a nonlinear ODE, for which we need to understand linear ODE first.

- Given a second order ODE, we can convert it to a first order system:

$$y''(t) + a\,y'(t) + b\,y(t) = 0$$

$$\frac{d}{dt}\begin{pmatrix} y(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} y'(t) \\ y''(t) \end{pmatrix} = \begin{pmatrix} y' \\ -ay'-by \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix}\begin{pmatrix} y \\ y' \end{pmatrix}$$

- Given a third order ODE, we can convert it to a first order system:

$$y'''(t) + a\,y''(t) + b\,y'(t) + c\,y(t) = 0$$

$$\frac{d}{dt}\begin{pmatrix} y \\ y' \\ y'' \end{pmatrix} = \begin{pmatrix} y' \\ y'' \\ y''' \end{pmatrix} = \begin{pmatrix} y' \\ y'' \\ -ay''-by'-cy \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -c & -b & -a \end{pmatrix}\begin{pmatrix} y \\ y' \\ y'' \end{pmatrix}$$

- So from now on, we focus on a first order ODE system

$$\frac{d}{dt}\vec{u}(t) = A\vec{u}(t), \quad \vec{u}(t) \in \mathbb{R}^n, \; A \in \mathbb{R}^{n\times n}$$

$$\frac{d}{dt}\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}$$

- The simplest case is $n=1$:

$$\frac{d}{dt}u(t) = a\,u(t)$$

$$\frac{u'(t)}{u(t)} = a$$

$$\int \frac{u'(t)}{u(t)}\,dt = \int a\,dt$$

$$\ln u(t) = at + C$$

$$\Rightarrow u(t) = e^{at} \cdot e^{c}$$

Plug in $t=0 \Rightarrow e^{c} = u(0)$

$$\Rightarrow u(t) = e^{at} u(0)$$

- $$\frac{d}{dt}\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - \lambda - 2 = (\lambda+1)(\lambda-2)$$

$\lambda = -1$ has eigenspace $\text{Span}\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$

$\lambda = 2$ has eigenspace $\text{Span}\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}$

$$V = \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix}, \quad D = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$$

$$V^{-1} = \frac{1}{-3}\begin{bmatrix} 1 & -2 \\ -1 & -1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix}$$

$$A = VDV^{-1} = \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix}^{-1}$$

$$\frac{d}{dt}\vec{u}(t) = A\vec{u}(t)$$

$$\frac{d}{dt}\vec{u}(t) = VDV^{-1}\vec{u}(t)$$

$$V^{-1}\frac{d}{dt}\vec{u}(t) = D V^{-1}\vec{u}(t)$$

$$\frac{d}{dt}\left[V^{-1}\vec{u}(t)\right] = D\cdot\left[V^{-1}\vec{u}(t)\right]$$

Change of variable $\vec{w}(t) = V^{-1}\vec{u}(t)$

$$\begin{pmatrix} w_1(t) \\ w_2(t) \end{pmatrix} = \frac{1}{3}\begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{3}u_1(t) + \frac{2}{3}u_2(t) \\ \frac{1}{3}u_1(t) + \frac{1}{3}u_2(t) \end{bmatrix}$$

$$\frac{d}{dt}\vec{w}(t) = D\vec{w}(t)$$

$$\begin{pmatrix} \frac{d}{dt}w_1(t) \\ \frac{d}{dt}w_2(t) \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} w_1(t) \\ w_2(t) \end{pmatrix}$$

$$\Rightarrow \begin{cases} \frac{d}{dt}w_1(t) = (-1)\cdot w_1(t) \\ \frac{d}{dt}w_2(t) = 2\cdot w_2(t) \end{cases}$$

$$\Rightarrow \begin{cases} w_1(t) = e^{-t}w_1(0) \\ w_2(t) = e^{2t}w_2(0) \end{cases}$$

$$\Rightarrow \vec{w}(t) = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{pmatrix}\vec{w}(0)$$

$\vec{w}(t) = V^{-1}\vec{u}(t)$

$$\Rightarrow \quad V^{-1}\vec{u}(t) = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{pmatrix} V^{-1}\vec{u}(0)$$

$$\Rightarrow \quad \vec{u}(t) = V \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{pmatrix} V^{-1}\vec{u}(0)$$

$At = tA = \begin{pmatrix} t & 2t \\ t & 0 \end{pmatrix}$ has eigenvalues $-t, 2t$

with same eigenvectors V.

$$\text{So} \quad e^{At} = V \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{pmatrix} V^{-1}$$

$$\Rightarrow \quad \vec{u}(t) = e^{At}\vec{u}(0)$$

$\frac{d}{dt}\vec{u}(t) = A\vec{u}(t)$

How matrix exponential is defined?

$$\text{If } A = VDV^{-1} = V \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} V^{-1}$$

$$\begin{aligned} \text{then } A^2 = A \cdot A &= VDV^{-1}VDV^{-1} \\ &= VD \cdot DV^{-1} \\ &= VD^2V^{-1} \\ &= V \begin{pmatrix} \lambda_1^2 & & \\ & \lambda_2^2 & \\ & & \lambda_3^2 \end{pmatrix} V^{-1} \end{aligned}$$

$$A^n = V \begin{pmatrix} \lambda_1^n & & \\ & \lambda_2^n & \\ & & \lambda_3^n \end{pmatrix} V^{-1}$$

We obtain e^A by

$$e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots$$

$$= VIV^{-1} + VDV^{-1} + \frac{1}{2!}VD^2V^{-1} + \frac{1}{3!}VD^3V^{-1} + \cdots$$

$$= V\left[I + D + \frac{1}{2!}D^2 + \frac{1}{3!}D^3 + \cdots\right]V^{-1}$$

$$V \begin{pmatrix} 1+\lambda_1+\frac{1}{2!}\lambda_1^2+\frac{1}{3!}\lambda_1^3+\cdots & 0 & 0 \\ 0 & 1+\lambda_2+\frac{1}{2!}\lambda_2^2+\frac{1}{3!}\lambda_2^3+\cdots & 0 \\ 0 & 0 & 1+\lambda_3+\frac{1}{2!}\lambda_3^2+\frac{1}{3!}\lambda_3^3+\cdots \end{pmatrix} V^{-1}$$

$$= V \begin{pmatrix} e^{\lambda_1} & & \\ & e^{\lambda_2} & \\ & & e^{\lambda_3} \end{pmatrix} V^{-1} \qquad e^A$$

$$A = VDV^{-1} = V \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} V^{-1}$$

$$tA = tVDV^{-1} = V \cdot tD \cdot V^{-1} = V \begin{pmatrix} -t & 0 \\ 0 & 2t \end{pmatrix} V^{-1}$$