The square matrix $A \in \mathbb{R}^{n\times n}$ is diagonalizable

$\Leftrightarrow$ $A = VDV^{-1}$ D is diagonal

$\Leftrightarrow$ A has n linearly independent eigenvectors

$\Leftrightarrow$ For any eigenvalue, the geo mul = alg mul

The following real matrices are always diagonalizable:

① Real Symmetric Matrix $A = A^T$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 6 & 7 \\ 3 & 6 & 8 & 9 \\ 4 & 7 & 9 & 10 \end{pmatrix}$$

② Real skew-Symmetric $A^T = -A$

$$\begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{pmatrix}$$

③ Normal Matrix $AA^T = A^TA$

Theorem If $A \in \mathbb{R}^{n\times n}$ satisfies $A = A^T$, then

① A has n real eigenvalues including repeated ones

② A has n orthogonal real eigenvectors, i.e., all eigenspaces are orthogonal to one another.

Example: $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

has 3 orthogonal (or orthonormal) eigenvectors

① $\det(\lambda I - A) = (\lambda+1)^2(\lambda-2)$

② Plug in $\lambda = -1$ into $(A - \lambda I)\vec{v} = \vec{0}$

Eigen-Space for $\lambda = -1$ is Span $\left\{ \overset{\vec{u}_1}{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}, \overset{\vec{u}_2}{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}} \right\}$

Plug in $\lambda = 2$ into $(A - \lambda I)\vec{v} = \vec{0}$

Eigen-Space for $\lambda = 2$ is Span $\left\{ \overset{\vec{u}_3}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}} \right\}$

③ To find orthogonal eigenvectors, apply Gram Schmit to basis vectors of each Eigen-Space with dimension ≥ 2

Apply Gram-Schmidt Procedure:

$$\vec{v}_1 = \vec{u}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{v}_1, \vec{u}_2 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1$$

$$= \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

$\boxed{\text{Verify } A\vec{v}_2 = (-1)\vec{v}_2}$

So we get orthogonal eigenvectors

$$\vec{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

and orthonormal eigenvectors:

$$\vec{w}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$\vec{w}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{pmatrix}$$

$$\vec{w}_2 = \frac{1}{\sqrt{\frac{3}{2}}} \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}\sqrt{\frac{2}{3}} \\ -\frac{1}{2}\sqrt{\frac{2}{3}} \\ \sqrt{\frac{2}{3}} \end{pmatrix}$$

$$\vec{w}_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{1}{3}} \\ \sqrt{\frac{1}{3}} \\ \sqrt{\frac{1}{3}} \end{pmatrix}$$

Why orthonormal eigenvectors:

Use $V = [\vec{w}_1 \ \vec{w}_2 \ \vec{w}_3]$ $\quad D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Then $A = VDV^{-1}$

and $V^{-1} = V^T \Rightarrow A = VDV^T$

$$V^T V = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Theorem If $A \in \mathbb{R}^{n\times n}$ is skew-symmetric ($A^T = -A^T$), then

① A has n purely imaginary eigenvalues including repeated ones

② A has n orthogonal complex eigenvectors, i.e., all eigenspaces are orthogonal to one another.

Example: $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $\lambda_1 = i$ $\vec{v}_1 = \begin{pmatrix} -i \\ 1 \end{pmatrix}$

$\begin{pmatrix} \cos\pi/2 & \sin\frac{\pi}{2} \\ -\sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{pmatrix}$ $\lambda_2 = -i$ $\vec{v}_2 = \begin{pmatrix} i \\ 1 \end{pmatrix}$

For two complex vectors $\vec{u}, \vec{v} \in \mathbb{C}^n$, the dot product is defined as

$$\langle \vec{u}, \vec{v} \rangle = \overline{\vec{v}}^T \vec{u} = [\overline{v_1} \ \overline{v_2} \ \overline{v_3}] \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$= \sum_{i=1}^{n} u_i \overline{v_i}.$$

So $\langle \vec{u}, \vec{v} \rangle = \overline{\langle \vec{v}, \vec{u} \rangle}$ $\vec{v}_1 = \begin{pmatrix} -i \\ 1 \end{pmatrix}$

We define $\vec{u} \perp \vec{v}$ as $\langle \vec{u}, \vec{v} \rangle = 0$. $\vec{v}_2 = \begin{pmatrix} i \\ 1 \end{pmatrix}$

$$\langle \vec{v}_1, \vec{v}_2 \rangle = \overline{\vec{v}_2}^T \vec{v}_1 = [-i \ \ 1] \begin{bmatrix} -i \\ 1 \end{bmatrix} = (-i)^2 + 1 = 0.$$

$$V = \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix} \qquad V^{-1} = \frac{1}{-2i}\begin{pmatrix} 1 & -i \\ -1 & -i \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = V \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} V^{-1}$$

Theorem If $A \in \mathbb{R}^{n\times n}$ is normal $(AA^T = A^TA)$ then

① A has n complex eigenvalues including repeated ones

② A has n orthogonal complex eigenvectors, i.e., all eigenspaces are orthogonal to one another.

Example: $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ is normal

$$A^TA = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$AA^T = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Ex: True or False

① If A is real symmetric, then A is normal.

② If A is skew-symmetric, then A is normal.

$$A^T = -A \Rightarrow \begin{cases} AA^T = A\cdot(-A) = -A^2 \\ A^TA = (-A)\cdot A = -A^2 \end{cases}$$

---

① Ordinary Differential Equation (ODE)

$$y''(t) + y(t) = 0$$

has infinitely many solutions, all of which form an abstract vector space

$$V = \{ y(t) : y''(t) + y(t) = 0 \}$$

Check the closedness of two operations:

1) If $y(t)$ is a sol, $a\cdot y(t)$ is also a sol $\forall a \in \mathbb{R}$.

2) If $x(t)$ and $z(t)$ are sol, $x(t) + z(t)$ is a sol.

② The dimension of the solution space is the order of ODE highest order of derivative

③ Given an n-th order ODE, we can rewrite it as first order system $\vec{u}'(t) = A\vec{u}(t)$

Ex: $y''(t) + y(t) = 0$

$$\vec{u}(t) = \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix} \Rightarrow \frac{d}{dt}\vec{u}(t) = \begin{pmatrix} y' \\ y'' \end{pmatrix} = \begin{pmatrix} y' \\ -y \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} y \\ y' \end{pmatrix}$$

$A \in \mathbb{R}^{n \times n}$

④ If $A\vec{v} = \lambda\vec{v}$, then $\vec{u}(t) = e^{\lambda t}\vec{v}$ is a sol to $\vec{u}'(t) = A\vec{u}(t)$

$$LHS = \frac{d}{dt}(e^{\lambda t}\vec{v}) = \frac{d}{dt}(e^{\lambda t})\vec{v} = \lambda e^{\lambda t}\vec{v}$$

$$RHS = A(e^{\lambda t}\vec{v}) = e^{\lambda t}(A\vec{v}) = e^{\lambda t}(\lambda\vec{v})$$

$\tilde{V} = \{\vec{u}(t) : \vec{u}'(t) = A\vec{u}(t)\}$ is another abstract vector space corresponding to V

⑤ If $\vec{v}_1, \cdots, \vec{v}_n$ are independent eigenvectors

$\lambda_1, \cdots, \lambda_n$ are corresponding eigenvalues

then we have $\vec{u}_i(t) = e^{\lambda_i t}\vec{v}_i$, $i = 1, 2, \cdots, n$

They are independent abstract vectors.

$$a_1 \vec{u}_1(t) + a_2 \vec{u}_2(t) + \cdots + a_n \vec{u}_n(t) = \vec{0}$$

$$\Rightarrow a_1 \vec{u}_1(0) + a_2 \vec{u}_2(0) + \cdots + a_n \vec{u}_n(0) = \vec{0}$$

$$\Rightarrow a_1 \vec{v}_1 + a_2 \vec{v}_2 + \cdots + a_n \vec{v}_n = \vec{0}$$

$$\Rightarrow a_1 = a_2 = \cdots = a_n = 0$$

⑥ A is diagonalizable $\Rightarrow$ All basis (solution) vectors are given by $e^{\lambda_i t}\vec{v}_i$

Any sol is a linear combination of $e^{\lambda_i t}\vec{v}_i$

⑦ Initial Value Problem of ODE

$$(\text{IVP}) \begin{cases} y''(t) + y(t) = 0 \\ y(0) = 0, \quad y'(0) = 1 \end{cases}$$ has a unique sol.

$$\Leftrightarrow \begin{cases} \frac{d}{dt}\begin{pmatrix} y(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} y(t) \\ y'(t) \end{pmatrix} \\ \begin{pmatrix} y(0) \\ y'(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{cases}$$

$$\Leftrightarrow \begin{cases} \frac{d}{dt}\vec{u}(t) = A\vec{u}(t) \\ \vec{u}(0) = \vec{u}_0 \end{cases}$$

The sol is $\vec{u}(t) = a_1 e^{\lambda_1 t}\vec{v}_1 + a_2 e^{\lambda_2 t}\vec{v}_2$

$$\vec{u}(0) = \vec{u}_0 \Rightarrow a_1\vec{v}_1 + a_2\vec{v}_2 = \vec{u}_0$$

$$\Rightarrow \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \vec{u}_0$$

$$\Rightarrow \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = V^{-1} \vec{u}_0$$

$$\Rightarrow \vec{u}(t) = [\vec{v}_1 \ \vec{v}_2] \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

$$= [\vec{v}_1 \ \vec{v}_2] \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} [\vec{v}_1 \ \vec{v}_2]^{-1} \vec{u}(0)$$

$$= e^{At} \vec{u}(0)$$

Example: Solve the IVP

$$\begin{cases} y''(t) + y(t) = 0 \\ y(0) = 0, \quad y'(0) = 1 \end{cases}$$

Solution: ① $\begin{cases} \dfrac{d}{dt} \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix} \\ \begin{pmatrix} y(0) \\ y'(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{cases}$

② $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$\lambda_1 = i, \quad \vec{v}_1 = \begin{pmatrix} -i \\ 1 \end{pmatrix}$

$\lambda_2 = -i, \quad \vec{v}_2 = \begin{pmatrix} i \\ 1 \end{pmatrix}$

$$V = \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix} \qquad V^{-1} = \frac{1}{-2i} \begin{pmatrix} 1 & -i \\ -1 & -i \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = V \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} V^{-1}$$

$$\Rightarrow A = \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \frac{1}{-2i} \begin{pmatrix} 1 & -i \\ -1 & -i \end{pmatrix}$$

③ $\vec{u}(t) = e^{At}\vec{u}(0)$

$$= V \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} V^{-1}\vec{u}(0)$$

$$= \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} \frac{1}{-2i} \begin{pmatrix} 1 & -i \\ -1 & -i \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} \frac{1}{-2i} \begin{pmatrix} -i \\ -i \end{pmatrix}$$

$$= \frac{1}{-2i} \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -ie^{it} \\ -ie^{-it} \end{pmatrix}$$

$$= \frac{1}{-2i} \begin{pmatrix} -e^{it} + e^{-it} \\ -ie^{it} - ie^{-it} \end{pmatrix}$$

$$\vec{u}(t) = \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix} \Rightarrow y(t) = \frac{1}{-2i}\left[-e^{it} + e^{-it}\right]$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$\begin{cases} e^{it} = \cos t + i\sin t \\ ie^{it} = i\cos t - \sin t \\ e^{-it} = \cos(-t) + i\sin(-t) \end{cases}$$

$$\Rightarrow y(t) = \frac{1}{-2i}\left[-\cos t - i\sin t + i(\cos t - \sin t)\right]$$

$$= \sin t$$

$$\begin{cases} y'' + y = 0 \\ y(0) = 0, \; y'(0) = 1 \end{cases}$$

Verify $y(t)$ satisfies
① ODE
② Initial Value