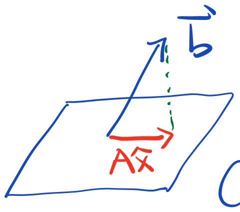
- Least Square, Projection

$A\vec{x} = \vec{b}$ $\quad A \in \mathbb{R}^{m\times n}$ $\quad \begin{array}{cc} & n \\ m & \boxed{A} \end{array}$ $\quad m > n$



$\hat{x}$ is the minimizer of $\|Ax-b\|^2$

$A\vec{x} = \vec{b}$ $\qquad \boxed{A}\,[\;] = [\;]$

$\Rightarrow A^TA\hat{x} = A^T\vec{b}$ $\qquad \boxed{A^T}\,\boxed{A} = \boxed{A^TA}$

Least Square Sol is $\hat{x} = (A^TA)^{-1}A^T\vec{b}$

Projection of $\vec{b}$ on Col(A) is $A\hat{x} = A(A^TA)^{-1}A^T\vec{b}$

① In order for $(A^TA)^{-1}$ to exist, A need to have independent column vectors.

If we want to project a vector $\vec{b}$ onto a subspace or a span, find a basis for that subspace. Use basis column vectors to form A.

② If $A = [\vec{v}_1 \;\; \vec{v}_2 \;\; \vec{v}_3]$ has orthogonal columns,

then $A^TA = \boxed{\;}\;\boxed{\;} = \begin{bmatrix} \|\vec{v}_1\|^2 & 0 & 0 \\ 0 & \|\vec{v}_2\|^2 & 0 \\ 0 & 0 & \|\vec{v}_3\|^2 \end{bmatrix}$

$\Rightarrow$ The projection of $\vec{b}$ onto Col (A) is also

$[A(A^TA)^{-1}A^T\vec{b} =]$ $\dfrac{\langle \vec{b}, \vec{v}_1\rangle}{\|\vec{v}_1\|^2}\vec{v}_1 + \dfrac{\langle \vec{b}, \vec{v}_2\rangle}{\|\vec{v}_2\|^2}\vec{v}_2 + \dfrac{\langle \vec{b}, \vec{v}_3\rangle}{\|\vec{v}_3\|^2}\vec{v}_3$

Example: Projection of $\vec{b}$ onto Span$\{\vec{v}\}$ is $\dfrac{\langle \vec{b}, \vec{v}\rangle}{\|\vec{v}\|^2}\vec{v}$

③ If $A = [\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3]$ has orthonormal columns,

then $A^TA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\Rightarrow$ The projection of $\vec{b}$ onto Col(A) is also

$[A(A^TA)^{-1}A^T\vec{b} =]$ $\langle \vec{b}, \vec{v}_1 \rangle \vec{v}_1 + \langle \vec{b}, \vec{v}_2 \rangle \vec{v}_2 + \langle \vec{b}, \vec{v}_3 \rangle \vec{v}_3$

- Gram-Schmit for generating orthogonal vectors

Question: Given Span$\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \cdots, \vec{v}_7\}$, how to find an orthogonal basis?

Answer: 1) Find a basis, say, $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

2) Apply Gram-Schmit to $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

$$\vec{u}_1 = \vec{v}_1$$

$$\vec{u}_2 = \vec{v}_2 - \frac{\langle \vec{v}_2, \vec{u}_1 \rangle}{\|\vec{u}_1\|^2} \vec{u}_1$$

$$\vec{u}_3 = \vec{v}_3 - \frac{\langle \vec{v}_3, \vec{u}_1 \rangle}{\|\vec{u}_1\|^2} \vec{u}_1 - \frac{\langle \vec{v}_3, \vec{u}_2 \rangle}{\|\vec{u}_2\|^2} \vec{u}_2$$

- Solutions to $\frac{d}{dt}\vec{u}(t) = A\,\vec{u}(t)$

$$\frac{d}{dt}\begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}\begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{bmatrix}$$

① Find eigenvalues λ_i with eigenvectors $\vec{v}_i$ of the matrix

② $\vec{u}(t) = e^{\lambda_i t}\vec{v}_i$ is one solution

③ If $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ are independent eigenvectors

then $e^{\lambda_1 t}\vec{v}_1$, $e^{\lambda_2 t}\vec{v}_2$, $e^{\lambda_3 t}\vec{v}_3$ are

independent solutions (abstract vectors)

④ If we have three independent eigenvectors, then

any solution $\vec{u}(t)$ can be written as

$$\vec{u}(t) = a_1 e^{\lambda_1 t}\vec{v}_1 + a_2 e^{\lambda_2 t}\vec{v}_2 + a_3 e^{\lambda_3 t}\vec{v}_3$$

⑤ Even if we do NOT three independent eigenvectors,

the dimension of the abstract vector space

(Solution Space) $V = \{\vec{u}(t) : \frac{d}{dt}\vec{u}(t) = A\vec{u}(t)\}$

is always 3.

⑥ (IVP)

initial value Problem

$$\begin{cases} \dfrac{d}{dt}\begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}\begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{bmatrix} \\ \begin{bmatrix} u_1(0) \\ u_2(0) \\ u_3(0) \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \end{cases}$$

Assume $A = VDV^{-1}$, then

$$At = V[Dt]V^{-1}$$

$$= V\begin{bmatrix} \lambda_1 t & 0 & 0 \\ 0 & \lambda_2 t & 0 \\ 0 & 0 & \lambda_3 t \end{bmatrix} V^{-1}$$

The solution to IVP is

$$\vec{u}(t) = e^{At}\,\vec{u}(0)$$

$$= V \begin{bmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 \\ 0 & 0 & e^{\lambda_3 t} \end{bmatrix} V^{-1}\,\vec{u}(0)$$