$A\vec{v} \in \mathbb{R}^2 \quad \vec{v} \in \mathbb{R}^3 \rightsquigarrow A \quad 2\times 3$

Matrix form of linear systems

$$\begin{cases} x + 2y + 3z = 1 \\ 3x + y + z = 0 \\ x + y + z = 0 \end{cases} \qquad A\vec{x} = \vec{b}$$

$$\underbrace{\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}}_{A \text{ coef matrix}} \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\vec{x}\ \text{unknown vector}} = \begin{pmatrix} 1\cdot x + 2\cdot y + 3\cdot z \\ 3x + y + z \\ x + y + z \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{\vec{b}\ \text{RHS vector}}$$

Ex: $\begin{cases} x+3y+z=0 \\ -x-y=1 \end{cases} \rightsquigarrow (-1)x+(-1)y+0\cdot z=1$

$A \quad 2\times 3$

$A=\begin{pmatrix} 1 & 3 & 1 \\ -1 & -1 & 0 \end{pmatrix} \qquad \vec{b}=\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$A\vec{x}=\vec{b} \qquad \vec{x}=\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

Augmented matrix

$(A\,|\,\vec{b}) \overset{Ex}{=\!=} \left(\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ -1 & -1 & 0 & 1 \end{array}\right)$

$\begin{cases} x-y=1 & (1) \\ 3x+2y=2 & (2) \end{cases}$

$2\cdot(1)+(2)$

$\left.\begin{array}{l} 2x-2y=2 \\ 3x+2y=2 \end{array}\right\} + \quad 5x=4 \Rightarrow x=\frac{4}{5}$

Gaussian Elimination for augmented matrix.

Idea: take linear combinations of equations / rows to simplify the problem

3 row operations (to the augmented matrix)

Type I row op: switch two rows

— — II — — : multi. a row by a nonzero scalar.

— — III — — — : multiply a row by a scalar then add it to another row.

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 3 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right) \xrightarrow[r_1 \leftrightarrow r_2]{\text{type I}} \left(\begin{array}{ccc|c} 3 & 1 & 1 & 0 \\ 1 & 2 & 3 & 1 \\ 1 & 1 & 1 & 0 \end{array}\right)$$

$$\xrightarrow[\frac{1}{3}r_1 \to r_1]{\text{type II}}$$

$$\left(\begin{array}{ccc|c} 1 & \frac{1}{3} & \frac{1}{3} & 0 \\ 1 & 2 & 3 & 1 \\ 1 & 1 & 1 & 0 \end{array}\right) \xrightarrow[(-1)r_1+r_3 \to r_3]{\text{type III}} \left(\begin{array}{ccc|c} 1 & \frac{1}{3} & \frac{1}{3} & 0 \\ 1 & 2 & 3 & 1 \\ \boxed{0} & \frac{2}{3} & \frac{2}{3} & 0 \end{array}\right)$$

$$(-1)r_1 + r_3 = r_3 - r_1$$

General process by an example:

goal: arrive at $\left(\begin{array}{ccc|c} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{array}\right)$ $\quad I_n\vec{x} = \vec{b}$, $I_n\vec{x} = \vec{x}$

subgoal: ——— $\left(\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \end{array}\right)$

Ex: $\begin{cases} x + 2y - z = 0 \\ 2y - z = 1 \\ x + 3z = -1 \end{cases}$

step 0: get the augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 2 & -1 & 1 \\ 1 & 0 & 3 & -1 \end{array}\right)$$

step 1: start with the 1st nonzero column

$$\begin{array}{c} \boxed{1} \\ * \\ * \\ * \end{array}$$

(a) do type I op. so that the 1st row in this column is nonzero

(b) then do type II op. so that this number $= 1$.

$$\boxed{\begin{matrix}1\\0\\0\\0\\0\end{matrix}}$$

step 2: Do type III op. to eliminate all other nonzero entries in this column

$r_1 - r_3 \to r_1$

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 2 & -1 & 1 \\ 1 & 0 & 3 & -1 \end{array}\right) \xrightarrow{r_3 - r_1 \to r_3} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 2 & -1 & 1 \\ 0 & -2 & 4 & -1 \end{array}\right)$$

step 3: do similar things to the submatrix (ignoring the previous column)

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 2 & -1 & 1 \\ 0 & -2 & 4 & -1 \end{array}\right) \xrightarrow[\text{type II}]{\frac{1}{2}r_2 \to r_2} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} \\ 0 & -2 & 4 & -1 \end{array}\right)$$

type III

$2r_2 + r_3 \to r_3$

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & \boxed{1} & -\frac{1}{2} & \frac{1}{2} \\ 0 & \boxed{0} & \boxed{3} & \boxed{0} \end{array}\right) \xrightarrow[\frac{1}{3}r_3 \to r_3]{\text{type } \mathrm{II}} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 0 \end{array}\right)$$

our subgoal.

final step: 2 options

(1) get $\left(\begin{array}{ccc|c} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{array}\right)$ by more types III operations.

(2) solve by substitution (work from bottom to top)

$r3 \Rightarrow \quad z = 0$

$r2 \Rightarrow \quad y - \frac{1}{2}z = \frac{1}{2} \xrightarrow{z=0} y = \frac{1}{2}$

$r1 \Rightarrow \quad x + 2y - z = 0 \xrightarrow{\begin{cases} y = \frac{1}{2} \\ z = 0 \end{cases}} x = -1$