The following $\boxed{\text{real}}$ matrices are always diagonalizable with orthogonal/orthonormal eigenvectors

① Real Symmetric Matrix $A = A^T$

$A = A^T \Rightarrow$ all eigenvalues are real $\Rightarrow$ real orthogonal eigenvectors

② Real Skew-Symmetric $A^T = -A$

$A = A^T \Rightarrow$ all eigenvalues are imaginary $\Rightarrow$ complex eigenvectors

③ Normal Matrix $AA^T = A^TA$ $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ is normal

Def: If $\vec{x}, \vec{y} \in \mathbb{C}^n$, then the dot product is

$$\langle \vec{x}, \vec{y} \rangle = \vec{y}^* \vec{x} = \overline{\vec{y}}^T \vec{x}$$

$*$ denotes conjugate transpose $\begin{bmatrix} i \\ 1 \end{bmatrix}^* = \begin{bmatrix} -i & 1 \end{bmatrix}$

Ex:
$$\left\langle \begin{bmatrix} 1+i \\ 1-i \end{bmatrix}, \begin{bmatrix} i \\ 1 \end{bmatrix} \right\rangle = \begin{bmatrix} \bar{i} & \bar{1} \end{bmatrix} \begin{bmatrix} 1+i \\ 1-i \end{bmatrix}$$
$$= \begin{bmatrix} -i & 1 \end{bmatrix} \begin{bmatrix} 1+i \\ 1-i \end{bmatrix}$$
$$= -i(1+i) + 1 \cdot (1-i)$$
$$= -i + 1 + 1 - i = 2 - 2i$$

Ex: $B = \begin{bmatrix} i & -i \\ -i & i \end{bmatrix}$ is not real symmetric

But still diagonalizable: $B = i \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$
$$= i\, V D V^{-1}$$
$$= V (iD) V^{-1}$$

Theorem: If $A \in \mathbb{C}^{n\times n}$ satisfies $A^* A = AA^*$ (normal),
$\bar{A}^T A = A\bar{A}^T$

then A is diagonalizable with orthonormal eigenvectors.

Remark: If eigenvectors are complex, they are orthonormal in the sense of zero complex dot product.

Example: $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $\lambda_1 = i$ $\vec{v}_1 = \begin{pmatrix} -i \\ 1 \end{pmatrix}$

$\lambda_2 = -i$ $\vec{v}_2 = \begin{pmatrix} i \\ 1 \end{pmatrix}$

$$\langle \vec{v}_1, \vec{v}_2 \rangle = \bar{\vec{v}}_2^T \vec{v}_1 = [-i \quad 1]\begin{bmatrix} -i \\ 1 \end{bmatrix} = (-i)^2 + 1 = 0.$$

Orthornormal eigenvectors

$$\vec{w}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \frac{1}{\sqrt{2}}\begin{pmatrix} -i \\ 1 \end{pmatrix}, \quad \vec{w}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|} = \frac{1}{\sqrt{2}}\begin{pmatrix} i \\ 1 \end{pmatrix}$$

$$V = \frac{1}{\sqrt{2}}\begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}, \quad V^{-1} = V^* = (\bar{V})^T = \frac{1}{\sqrt{2}}\begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = V \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} V^*$$

- Eigenvalue Decomposition:

If $A \in \mathbb{R}^{n\times n}$ is diagonalizable, then

$$A = VDV^{-1} = V \begin{bmatrix} d_1 & & \\ & d_2 & \\ & & d_3 \end{bmatrix} V^{-1}$$

$$\Leftrightarrow \quad AV = VD$$

$$\Leftrightarrow \quad A[\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3] = [\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3] \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$$

$$\Leftrightarrow [A\vec{v}_1 \ A\vec{v}_2 \ A\vec{v}_3] = [d_1\vec{v}_1 \ d_2\vec{v}_2 \ d_3\vec{v}_3]$$

$$\Leftrightarrow \quad A\vec{v}_i = d_i \vec{v}_i$$

$\Leftrightarrow$ $\vec{v}_i$ are independent eigenvectors with eigen-values d_i

Remark: d_i & $\vec{v}_i$ might be complex

- The following are always diagonalizable with orthonormal eigenvectors:

① complex normal matrix ($AA^* = A^*A, A \in \mathbb{C}^{n\times n}$) $A^* = \bar{A}^T$

② real normal matrix ($AA^T = A^TA$, $A \in \mathbb{R}^{n\times n}$)

③ real symmetric ($A^T = A$):
1) all eigenvalues are real
2) thus eigenvectors are real

$$\Rightarrow \quad A = VDV^{-1} = VDV^T$$

orthonormal cols in $V \Rightarrow V^TV = I$

④ real skew-symmetric ($A^T = -A$):
1) all eigenvalues are purely imaginary
2) thus eigenvectors are complex

$$\Rightarrow A = VDV^{-1} = VDV^*$$

complex orthonormal cols in $V \Rightarrow V^*V = I$

In general, A is normal $\Leftrightarrow A = VDV^*$.

- Singular Value Decomposition (SVD)

For any $A \in \mathbb{R}^{n\times n}$, there are $U, \Sigma, V \in \mathbb{R}^{n\times n}$ s.t.

$$A = U\Sigma V^T$$

① $\Sigma = \begin{pmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \end{pmatrix}$, $\sigma_i \geq 0$ are singular values

$$\sigma(A) = \sqrt{\lambda(A^TA)} = \sqrt{\lambda(AA^T)}$$

$\hookrightarrow$ eigenvalue

② Cols of U are orthonormal : left singular vectors

$$U^TU = UU^T = I$$

③ Cols of V are orthonormal : right singular vectors

$$V^TV = VV^T = I$$

④ $A = U\Sigma V^T$

$$\Leftrightarrow A = U\Sigma V^{-1}$$

$$\Leftrightarrow AV = U\Sigma$$

$$\Leftrightarrow A[\vec{v}_1\ \vec{v}_2\ \vec{v}_3] = [\vec{u}_1\ \vec{u}_2\ \vec{u}_3]\begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

$$\Leftrightarrow [A\vec{v}_1\ A\vec{v}_2\ A\vec{v}_3] = [\sigma_1\vec{u}_1\ \sigma_2\vec{u}_2\ \sigma_3\vec{u}_3]$$

$$\Leftrightarrow A\vec{v}_i = \sigma_i\vec{u}_i$$ The order must match

⑤ $A = U\Sigma V^T \Rightarrow AA^T = U\Sigma V^T V\Sigma U^T = U\Sigma^2 U^T$

Review: A is real positive definite if ① $A = A^T$ ② $\vec{x}^T A \vec{x} > 0$, $\forall \vec{x} \in \mathbb{R}^n$, $\vec{x} \neq \vec{0}$

- Theorem A real symmetric $A \in \mathbb{R}^{n\times n}$ is positive (semi-)definite if and only if all eigenvalues of A are positive. (non-negative)

$\Rightarrow \begin{cases} \vec{u}_i \text{ are orthonormal eigenvectors of } AA^T. \\ \sigma_i^2 \text{ are eigenvalues of } AA^T. \end{cases}$

$A = U\Sigma V^T \Rightarrow A^T A = V\Sigma U^T U\Sigma V^T = V\Sigma^2 V^T$

$\Rightarrow \begin{cases} \vec{v}_i \text{ are orthonormal eigenvectors of } A^TA. \\ \sigma_i^2 \text{ are eigenvalues of } A^TA. \end{cases}$

And we have pick right order and signs of $\vec{u}_i$, $\vec{v}_i$ s.t.

$$A\vec{v}_i = \sigma_i \vec{u}_i$$