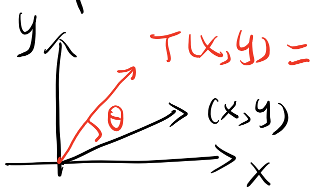
Linear Transformation $T: V \longrightarrow W$

$\vec{v} \longmapsto T(\vec{v})$

T preserves two operations
$$\begin{cases} ① \ T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v}), & \forall \vec{u}, \vec{v} \in V \\ ② \ T(a\vec{v}) = aT(\vec{v}), & \forall a \in \mathbb{R} \Rightarrow T(\vec{0}) = \vec{0} \end{cases}$$

Examples of Linear Transformation from $\mathbb{R}^2 \to \mathbb{R}^2$

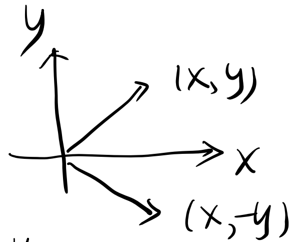
① Rotation



$T(x,y) = (\cos\theta\, x - \sin\theta\, y, \ \sin\theta\, x + \cos\theta\, y)$

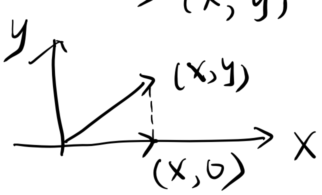
$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

②



Reflection $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

③



Projection $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

Ex 1: $V = C(\mathbb{R}) = \{$all continuous functions$\}$

$T: V \longrightarrow \mathbb{R}$

$f(x) \longmapsto \int_a^b f(t)\,dt$

Ex 2: $V = \{$all differentiable functions$\}$

$W = \{$all functions$\}$

$T: V \longrightarrow W$

$f(x) \longmapsto f'(x)$

Def ordered basis : a basis with an order.

Ex: $\mathbb{R}^3$, $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Ex : $P_n(\mathbb{R})$, $\{1, x, x^2, \cdots, x^n\}$

Def: Given an ordered basis $\beta = \{\vec{u}_1, \dots, \vec{u}_n\}$ of V

$\forall \vec{v} \in V$, $\vec{v} = a_1 \vec{u}_1 + \dots + a_n \vec{u}_n$

$[\vec{v}]_\beta = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ is the coordinate of $\vec{v}$ under basis β.

Ex: $P_2(\mathbb{R})$, $\beta = \{1, x, x^2\}$

$f(x) = 4 + 6x - 7x^2$

$[f]_\beta = \begin{bmatrix} 4 \\ 6 \\ -7 \end{bmatrix}$

Def $T: V \longrightarrow W$

$\beta = \{\vec{v}_1, \dots, \vec{v}_n\}$ is an ordered basis of V

$\gamma = \{\vec{w}_1, \dots, \vec{w}_m\}$ — — — — — W

Then $T(\vec{v}_j) = \sum_{i=1}^{m} a_{ij} \vec{w}_i$, $\forall j = 1, \dots, n$.

$A = [a_{ij}]_{m \times n}$ is the matrix representation of T under bases β and γ.

We write $A = [T]_\beta^\gamma$

$$A = \begin{bmatrix} & \boxed{\begin{matrix} a_{1j} \\ \vdots \\ a_{mj} \end{matrix}} & \end{bmatrix}_{m \times n}$$

$$\begin{matrix} \boxed{\begin{matrix} a_{1j} \\ \vdots \\ a_{mj} \end{matrix}} \\ \parallel \\ [T(\vec{v}_j)]_\gamma \end{matrix} \qquad T(\vec{v}_j) \in W.$$

Ex: $T: P_3(\mathbb{R}) \longrightarrow P_2(\mathbb{R})$ $\quad \beta = \{1, x, x^2, x^3\}$

$f(x) \longmapsto f'(x)$ $\quad \gamma = \{1, x, x^2\}$

$$T(1) = 0 = 0\cdot 1 + 0\cdot x + 0\cdot x^2$$
$$T(x) = 1 = 1\cdot 1 + 0\cdot x + 0\cdot x^2$$
$$T(x^2) = 2x = 0\cdot 1 + 2\cdot x + 0\cdot x^2$$
$$T(x^3) = 3x^2 = 0\cdot 1 + 0\cdot x + 3x^2$$

$$[T]_\beta^\gamma = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}_{3\times 4}$$

Ex: $I_V : V \longrightarrow V$, $\vec{v} \longmapsto \vec{v}$ $\qquad \beta = \{\vec{v}_1, \dots, \vec{v}_n\}$

$$[I_V]_\beta^\beta = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

Def $\quad T : V \longrightarrow V \qquad \beta = \{v_1, \dots, v_n\}$

$$[T]_\beta^\beta \in \mathbb{R}^{n\times n}$$

Remark: Given an n-dimensional V with an ordered basis β, any linear transformation $T : V \longrightarrow V$ corresponds to a square matrix $[T]_\beta^\beta$

Example:

Geometry $\longleftrightarrow$ Algebra

$V = \mathbb{R}^2 \qquad T : V \longrightarrow V$

$\beta = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

$\mathbb{R}^{2\times 2}$

① Rotation by 30° counterclockwise

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{pmatrix} \cos\frac{\pi}{6} & -\sin\frac{\pi}{6} \\ \sin\frac{\pi}{6} & \cos\frac{\pi}{6} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$[T]_\beta^\beta = \begin{pmatrix} \cos\frac{\pi}{6} & -\sin\frac{\pi}{6} \\ \sin\frac{\pi}{6} & \cos\frac{\pi}{6} \end{pmatrix}$$

② Reflection w.r.t. x-axis

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$[T]_\beta^\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

③ Projection to x-axis

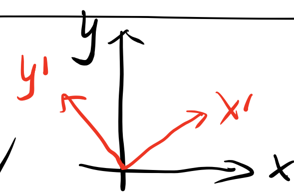
$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$[T]_\beta^\beta = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

④ Identity

$$[I]_\beta^\beta = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Change of Coordinates



Given $\left.\begin{matrix} \beta = \{\vec{v}_1, \cdots, \vec{v}_n\} \\ \gamma = \{\vec{w}_1, \cdots, \vec{w}_n\} \end{matrix}\right\}$ bases of V

Q: Given $[\vec{v}]_\beta$, what is $[\vec{v}]_\gamma$?

$$\begin{aligned} I :\ & V \longrightarrow V \\ & \vec{v} \longmapsto \vec{v} \end{aligned}$$

$$[\vec{v}]_\gamma = [I(\vec{v})]_\gamma = [I]_\beta^\gamma [\vec{v}]_\beta$$

$$\left(T : \underset{\beta}{\overset{n}{V}} \longrightarrow \underset{\gamma}{\overset{m}{W}} \qquad T(\vec{v}_j) = \sum_{i=1}^{m} a_{ij}\vec{w}_i \right)$$

$$\vec{v} \longmapsto T(\vec{v})$$

$$[T]_\beta^\gamma = \left(a_{ij} \right)_{m\times n}$$

$$[I]_\beta^\gamma = \Big(\quad \underset{j\text{-th col}}{\boxed{[I(\vec{v}_j)]_\gamma}} \quad \Big)_{n\times n} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

Ex: $P_2(\mathbb{R})$ $\quad \beta = \{1, x, x^2\}$

$\gamma = \{1, (x-1), (x-1)^2\}$

$$I(1) = 1 = 1\cdot 1 + 0\cdot(x-1) + 0\cdot(x-1)^2$$

$$I(x) = x = 1\cdot 1 + 1\cdot(x-1) + 0\cdot(x-1)^2$$

$$I(x^2) = x^2 = a\cdot 1 + b(x-1) + c(x-1)^2$$

Solve for $a, b, c \Rightarrow \begin{cases} a=1 \\ b=2 \\ c=1 \end{cases}$

$$= 1\cdot 1 + 2\cdot(x-1) + 1\cdot(x-1)^2$$

Given $f(x) = ax^2 + bx + c$

want $f(x) = A(x-1)^2 + B(x-1) + C$

$[f]_\beta = \begin{pmatrix} c \\ b \\ a \end{pmatrix}$, $\quad [f]_\gamma = \begin{bmatrix} C \\ B \\ A \end{bmatrix}$

$$[f]_\gamma = [I]_\beta^\gamma [f]_\beta$$

$$\begin{bmatrix} C \\ B \\ A \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c \\ b \\ a \end{bmatrix} = \begin{pmatrix} a+b+c \\ b+2a \\ a \end{pmatrix}$$

Now just focus on $L_A : \mathbb{R}^n \to \mathbb{R}^n$
$\vec{v} \mapsto A\vec{v}$
for a square matrix $A \in \mathbb{R}^n$

$\boxed{\text{Theorem}}$

Let $\beta = \{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\}$ be an ordered basis of $\mathbb{R}^n$

Let $V = [\vec{v}_1 \ \vec{v}_2 \ \cdots \ \vec{v}_n] \in \mathbb{R}^{n\times n}$.

For $L_A : \mathbb{R}^n \to \mathbb{R}^n$, its matrix representation under basis β is $[L_A]_\beta^\beta = V^{-1}AV$

Remark: If $V^{-1}AV = D$ is diagonal ($\Leftrightarrow A = VDV^{-1}$), then it means the matrix representation of the linear transformation L_A is diagonal under basis $\beta = \{\vec{v}_1, \cdots, \vec{v}_n\}$

$$\Downarrow$$
$$L_A(\vec{v}_i) = d_i\vec{v}_i$$

Question: how to find β if this is possible?

Answer: find n independent eigenvectors of A

Theorem

Let $\beta = \{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\}$ be an ordered basis of $\mathbb{R}^n$

$\gamma = \{\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_n\}$ be an ordered basis of $\mathbb{R}^n$

Let $V = [\vec{v}_1 \ \vec{v}_2 \ \cdots \ \vec{v}_n] \in \mathbb{R}^{n\times n}$

$U = [\vec{u}_1 \ \vec{u}_2 \ \cdots \ \vec{u}_n] \in \mathbb{R}^{n\times n}$

For $L_A : \underset{\beta}{\mathbb{R}^n} \to \underset{\gamma}{\mathbb{R}^n}$, its matrix representation under bases β and γ

is $[L_A]_\beta^\gamma = U^{-1}AV$

Corollary:

If $\beta = \{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\}$ are left singular vectors

$\gamma = \{\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_n\}$ are right singular vectors,

then $\underbrace{A = U\Sigma V^T}_{\text{SVD}} \Rightarrow [L_A]_\beta^\gamma = U^{-1}AV = \Sigma$

Matrix representation of L_A is diagonal!

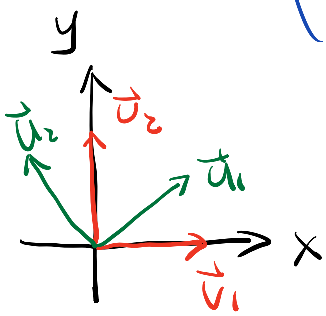
$L_A(\vec{v}_i) = \sigma_i \vec{u}_i$

Remark: ① Diagonalization $A = VDV^{-1}$ is not always possible

② SVD $A = U\Sigma V^T$ is always true!

Example: Rotation by $\theta = 30°$

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = U\Sigma V^T = \underbrace{\begin{pmatrix} \vec{u}_1 & \vec{u}_2 \end{pmatrix}}_{U} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\Sigma} \underbrace{\begin{pmatrix} \vec{v}_1^T \\ \vec{v}_2^T \end{pmatrix}}_{V^T}$$



$$L_A : \mathbb{R}^2 \to \mathbb{R}^2$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto A\begin{pmatrix} x \\ y \end{pmatrix} \quad \begin{cases} L_A(\vec{v}_1) = \vec{u}_1 \\ L_A(\vec{v}_2) = \vec{u}_2 \end{cases}$$

$$AA^T = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = A^T A$$

For this case, we can pick any orthonormal basis $\beta = \{\vec{v}_1, \vec{v}_2\}$

and define $\begin{cases} \vec{u}_1 = A\vec{v}_1 \\ \vec{u}_2 = A\vec{v}_2 \\ \gamma = \{\vec{u}_1, \vec{u}_2\} \end{cases}$

s.t. $[L_A]_\beta^\gamma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$