Review

Def: Given an ordered basis $\beta = \{\vec{u}_1, \dots, \vec{u}_n\}$ of V

$\forall \vec{v} \in V$, $\vec{v} = a_1 \vec{u}_1 + \dots + a_n \vec{u}_n$

$[\vec{v}]_\beta = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ is the coordinate of $\vec{v}$ under basis β.

Ex: $P_2(\mathbb{R})$, $\beta = \{1, x, x^2\}$

$f(x) = 4 + 6x - 7x^2$

$[f]_\beta = \begin{bmatrix} 4 \\ 6 \\ -7 \end{bmatrix}$

Def $T: V \longrightarrow W$

$\beta = \{\vec{v}_1, \dots, \vec{v}_n\}$ is an ordered basis of V

$\gamma = \{\vec{w}_1, \dots, \vec{w}_m\}$ - - - - - - - W

Then $T(\vec{v}_j) = \sum_{i=1}^{m} a_{ij} \vec{w}_i$, $\forall j = 1, \dots, n$.

$A = [a_{ij}]_{m \times n}$ is the matrix representation of T under bases β and γ.

We write $A = [T]_\beta^\gamma$

$$A = \begin{bmatrix} & \boxed{\begin{matrix} a_{1j} \\ \vdots \\ a_{mj} \end{matrix}} & \end{bmatrix}_{m \times n}$$

$\boxed{\begin{matrix} a_{1j} \\ \vdots \\ a_{mj} \end{matrix}} = [T(\vec{v}_j)]_\gamma$ $\quad T(\vec{v}_j) \in W$.

Ex: $T: P_3(\mathbb{R}) \longrightarrow P_2(\mathbb{R})$ $\quad \beta = \{1, x, x^2, x^3\}$

$f(x) \longmapsto f'(x)$ $\quad \gamma = \{1, x, x^2\}$

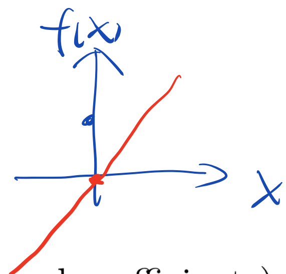
$T(1) = 0 = 0\cdot 1 + 0\cdot x + 0\cdot x^2$

$T(x) = 1 = 1\cdot 1 + 0\cdot x + 0\cdot x^2$

$T(x^2) = 2x = 0\cdot 1 + 2\cdot x + 0\cdot x^2$

$T(x^3) = 3x^2 = 0\cdot 1 + 0\cdot x + 3x^2$

$$[T]_\beta^\gamma = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}_{3\times 4}$$



HW#9

4. (10 pts) Let $V = P_2(\mathbb{R})$ (all quadratic polynomials with real coefficients) and consider a linear transformation $T : V \longrightarrow V$ defined as

$$T[f(x)] = f(0)x + f'(x) - \frac{1}{2}f''(x).$$

$f(x) = 1 \Rightarrow f(0) = 1$

For the ordered basis $\beta = \{1, x, x^2\}$, find the matrix representation $[T]_\beta^\beta$ of T under basis β.

3×3

β $\qquad$ $\gamma = \beta$

$T : P_2(\mathbb{R}) \longrightarrow P_2(\mathbb{R})$

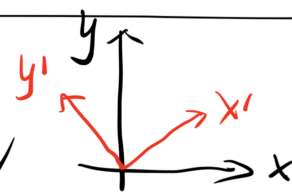
$f(x) = ax^2 + bx + c \longmapsto$

$T[x] = T[f(x)] = 0\cdot x + 1 - \frac{1}{2}\cdot 0 = 1$

$$f(x) = x \Rightarrow \begin{cases} f(0) = 0 \\ f'(x) = 1 \\ f''(x) = 0 \end{cases}$$

Change of Coordinates

Given $\left.\begin{array}{l} \beta = \{\vec{v}_1, \dots, \vec{v}_n\} \\ \gamma = \{\vec{w}_1, \dots, \vec{w}_n\} \end{array}\right\}$ bases of V



Q: Given $[\vec{v}]_\beta$, what is $[\vec{v}]_\gamma$?

$$I : V \longrightarrow V$$
$$\vec{v} \longmapsto \vec{v}$$

$$[\vec{v}]_\gamma = [I(\vec{v})]_\gamma = [I]_\beta^\gamma [\vec{v}]_\beta$$

$[I]_\beta^\gamma$ is the change of coordinate matrix

$[I(\vec{v}_j)]_\gamma$

$$[I]_\beta^\gamma = \begin{pmatrix} & \boxed{} & \end{pmatrix}_{n\times n} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

j-th col

Ex: $P_2(\mathbb{R})$ $\quad \beta = \{1, x, x^2\}$

$\quad \gamma = \{1, (x-1), (x-1)^2\}$

$$I(1) = 1 = 1\cdot 1 + 0\cdot(x-1) + 0\cdot(x-1)^2$$
$$I(x) = x = 1\cdot 1 + 1\cdot(x-1) + 0\cdot(x-1)^2$$
$$I(x^2) = x^2 = a\cdot 1 + b(x-1) + c(x-1)^2$$

Solve for $a, b, c \Rightarrow \begin{cases} a=1 \\ b=2 \\ c=1 \end{cases}$

$[I]_\beta^\gamma$

$$= 1\cdot 1 + 2\cdot(x-1) + 1\cdot(x-1)^2$$

Given $f(x) = ax^2 + bx + c$

want $f(x) = A(x-1)^2 + B(x-1) + C$

$$[f]_\beta = \begin{pmatrix} c \\ b \\ a \end{pmatrix}, \quad [f]_\gamma = \begin{bmatrix} C \\ B \\ A \end{bmatrix}$$

$$[f]_\gamma = [I]_\beta^\gamma [f]_\beta$$

$$\begin{bmatrix} C \\ B \\ A \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c \\ b \\ a \end{bmatrix} = \begin{pmatrix} a+b+c \\ b+2a \\ a \end{pmatrix}.$$

HW #9

5. (10 pts) Let V be the set consisting all continuous real-valued single-variable functions. Then V is a vector space. Consider a subspace $W = span\{1, \sin x, \cos x, \sin(2x), \cos(2x)\}$ with two ordered bases of W:

$$\beta = \{1, -\cos x, \sin x, \sin(2x), \sin^2 x\}$$

$$\gamma = \{1, \sin x, \cos x, \sin(2x), \cos^2 x\}.$$

Find the change of coordinate matrix from β to γ, i.e., the matrix Q s.t. $[f]_\gamma = Q[f]_\beta, \forall f \in W$. Recall that Q is the matrix reprensetation $[I]_\beta^\gamma$ for the identity map under bases β and γ.

$$I : W \longrightarrow W$$

Find $[I]_\beta^\gamma = \begin{bmatrix} \boxed{} & \boxed{} & \boxed{} & \boxed{} & \boxed{} \end{bmatrix}$

$$I(\sin x) = \sin x = 0 \cdot 1 + 1 \cdot \sin x + 0 \cdot \cos x + 0 \cdot \sin(2x) + 0 \cdot \cos^2 x$$

Now just focus on $L_A : \mathbb{R}^n \to \mathbb{R}^n$
$\vec{v} \mapsto A\vec{v}$

for a square matrix $A \in \mathbb{R}^n$

Theorem

Let $\beta = \{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\}$ be an ordered basis of $\mathbb{R}^n$

Let $V = [\vec{v}_1 \; \vec{v}_2 \; \cdots \; \vec{v}_n] \in \mathbb{R}^{n\times n}$.

For $L_A : \mathbb{R}^n \to \mathbb{R}^n$, its matrix representation under basis β is $[L_A]_\beta^\beta = V^{-1}AV$

In particular, if $\vec{v}_i$ are eigenvectors ($\Leftrightarrow$ A is diagonalizable)
($\vec{v}_i$ are basis thus independent)

then $A = V \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} V^{-1}$

$\Rightarrow [L_A]_\beta^\beta = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$

So the matrix representation of L_A is diagonal under a basis consisting of eigenvectors

Example: If $A \in \mathbb{R}^{3\times3}$ is diagonalizable

then we can find independent eigenvectors

β $\underline{\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}}$ which is a basis of $\mathbb{R}^3$

So any $\vec{v} \in \mathbb{R}^3$ can be written as

$$[\vec{v}]_\beta = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \qquad \vec{v} = a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3$$

and

$$A\vec{v} = A[a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3]$$

$$[A\vec{v}]_\beta = \begin{bmatrix} \lambda_1 a_1 \\ \lambda_2 a_2 \\ \lambda_3 a_3 \end{bmatrix} \qquad = a_1 A\vec{v}_1 + a_2 A\vec{v}_2 + a_3 A\vec{v}_3$$

$$= a_1\lambda_1\vec{v}_1 + a_2\lambda_2\vec{v}_2 + a_3\lambda_3\vec{v}_3$$

Generalized Eigenvector

Not all matrices are diagonalizable.

For non-diagonalizable matrices,

① Not enough independent eigenvectors

② There is one eigenvalue for which

geometrical mul < algebraic mul

③ Let λ be this eigenvalue s.t.

geo mul < alg mul

Let $\vec{v}$ be one eigenvector.

Then $(\lambda I - A)\vec{u} = \vec{v}$ has a sol

and this solution $\vec{u}$ is called

generalized eigenvector

HW#9

2. (30 pts) Consider the matrix

$$A = \begin{bmatrix} 3 & -1 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}.$$

(a) (10 pts) Find all eigenvalues, their algebraic multiplicity and geometrical multiplicity, and basis vectors for all eigenspaces.

(b) (10 pts) For this particular matrix, there is one eigenvalue λ_2 for which geometrical multiplicity is less than algebraic multiplicity. This ensures existence of one *generalized eigenvector* defined as follows: let v be its eigenvector, then find the generalized eigenvector u defined as solution to the nonhomogeneous linear system

$$(A - \lambda_2 I)u = v. \iff A\vec{u} - \lambda_2 \vec{u} = \vec{v}$$

(c) (10 pts) For this particular matrix, there are two distinct eigenvalues λ_1 and λ_2. Let v_1 be eigenvector for λ_1. Form a matrix $V = [v_1 \; v \; u]$. Then by the definition of eigenvectors and generalized eigenvectors, we have

$$AV = [Av_1 \; Av \; Au] = [\lambda_1 v_1 \; \lambda_2 v \; \lambda_2 u + v] = [v_1 \; v \; u] \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 1 \\ 0 & 0 & \lambda_2 \end{bmatrix}.$$

Here $J = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 1 \\ 0 & 0 & \lambda_2 \end{bmatrix}$ is called Jordan Form of A. Find the explicit expression of J, V, V^{-1} and verify that $A = VJV^{-1}$ (and this is what eigenvalue decomposition looks like for a nondiagonalizable matrix).

The following are NOT required

Jordan Normal Form (FYI, not required)

Def Jordan block for eigenvalue λ

$$J_k(\lambda) = \begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}_{k \times k}$$

$$(\lambda) \qquad \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \qquad \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} \qquad \begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix}$$

Theorem For any $A \in \mathbb{R}^{n \times n}$, we can find n linearly independent eigenvectors & generalized eigenvectors $\vec{v}_1, \cdots, \vec{v}_n$ such that

$$A = VJV^{-1} \quad \text{where } V = [\vec{v}_1 \cdots \vec{v}_n]$$

$$J = \begin{pmatrix} \boxed{J_{k_1}(\lambda_1)} & 0 & 0 & 0 \\ 0 & \boxed{J_{k_2}(\lambda_2)} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \boxed{J_{k_\ell}(\lambda_\ell)} \end{pmatrix}$$

Example: If $A = VJV^{-1} \in \mathbb{R}^{6\times 6}$ and $\det(\lambda I - A) = (\lambda+1)^3(\lambda-2)^3$

$$J = \begin{pmatrix} \boxed{-1} & 0 & 0 \\ 0 & \boxed{\begin{matrix} -1 & 1 \\ 0 & -1 \end{matrix}} & 0 \\ 0 & 0 & \boxed{\begin{matrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{matrix}} \end{pmatrix}$$

eigenvalues	$\lambda_1 = -1$	$\lambda_2 = 2$
alg mul	$1+2=3$	3
geo mul	$1+1$	1

\# of blocks

$$J_k(\lambda) = \lambda I_{k\times k} + J_k$$

$$\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} = \lambda I + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$J_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$J_3^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$J_3^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

We can use $A = PJP^{-1}$

to compute A^n and e^A

$$A^n = PJ^nP^{-1}$$

$$= P \begin{pmatrix} \boxed{J_1^n} & & & \\ & \boxed{J_2^n} & & \\ & & \ddots & \\ & & & \boxed{J_k^n} \end{pmatrix}$$

$$\frac{d}{dt} \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{bmatrix} = \begin{bmatrix} & A & \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{bmatrix}$$

First Order Homogeneous Linear ODE Systems: General Case

- In general, the matrix has eigenvalue decomposition as $A = QJQ^{-1}$ where columns of Q are eigenvectors and generalized eigenvectors, and J is the Jordan Form.
- The solution formula for $\mathbf{u}' = A\mathbf{u}$ is still $\mathbf{u} = e^{At}\mathbf{u}(0)$.
- The matrix exponential is still defined as the Maclaurin Series [plug in $At = Q(Jt)Q^{-1}$]:

$$e^{At} = I + A + \frac{1}{2!}A^2 + \cdots = Q(I + Jt + \frac{1}{2!}[Jt]^2 + \cdots)Q^{-1}$$

- If there are k distinct eigenvalues, J is block diagonal with k blocks $\Lambda_1, \cdots \Lambda_k$.
- The matrix exponential e^J (or e^{Jt}) will also be block diagonal with k blocks e^{Λ_i} (or $e^{\Lambda_i t}$).
- If Λ_i is diagonal (i.e., geo multiplicity=algebraic multiplicity for λ_i), then e^J (or e^{Jt}) is diagonal.

$$\vec{u}(t) = Q\,\underline{e^{Jt}}\,Q^{-1}\vec{u}(0)$$

- In general, e^J (or e^{Jt}) are upper triangular.

$$\begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{bmatrix} \qquad \begin{bmatrix} e^{\lambda_1 t} & te^{\lambda_1 t} & \frac{t^2}{2}e^{\lambda_1 t} \\ 0 & e^{\lambda_1 t} & te^{\lambda_1 t} \\ 0 & 0 & e^{\lambda_1 t} \end{bmatrix} = e^{\lambda_1 t}\begin{bmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$$

A Quick Example

- Consider solving $\mathbf{u}'(t) = A\mathbf{u}(t)$ with a 3×3 matrix A.
- Assume A has only one distinct eigenvalue λ, and only one linearly independent eigenvector $\mathbf{v}$.
- And further assume there are two linearly independent generalized eigenvectors $\mathbf{u}_1$ and $\mathbf{u}_2$. Namely, $\mathbf{u}_1$ and $\mathbf{u}_2$ satisfies

$$A\mathbf{v} = \lambda \mathbf{v} \tag{1}$$

$$A\mathbf{u}_1 = \lambda \mathbf{u}_1 + \mathbf{v} \tag{2}$$

$$A\mathbf{u}_2 = \lambda \mathbf{u}_2 + \mathbf{u}_1 \tag{3}$$

$$\vec{u}(t) = e^{At}\vec{u}(0)$$

$$= Q e^{\lambda t} \begin{pmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} Q^{-1}\vec{u}(0)$$

- Let $Q = [\mathbf{v}\ \mathbf{u}_1\ \mathbf{u}_2]$, then $A = \underbrace{Q \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} Q^{-1}}$. Do change of variable $\mathbf{w} = Q^{-1}\mathbf{u}$.
- We can verify the unique solution has the form:

$$\frac{d}{dt}\begin{pmatrix} w_1(t) \\ w_2(t) \\ w_3(t) \end{pmatrix} = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}\begin{pmatrix} w_1(t) \\ w_2(t) \\ w_3(t) \end{pmatrix} \Leftrightarrow \begin{pmatrix} w_1(t) \\ w_2(t) \\ w_3(t) \end{pmatrix} = e^{\lambda t}\begin{pmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} w_1(0) \\ w_2(0) \\ w_3(0) \end{pmatrix}$$

Independent Solutions to ODE are

① $e^{\lambda t}\vec{v}$

② $e^{\lambda t}\vec{u}_1 + te^{\lambda t}\vec{v}$

③ $e^{\lambda t}\vec{u}_2 + te^{\lambda t}\vec{u}_1 + \frac{t^2}{2}e^{\lambda t}\vec{v}$