Review

- A matrix of size $m \times n$ has $\left\{ \begin{array}{l} mn \text{ entries} \\ m \text{ rows} \\ n \text{ cols} \end{array} \right.$

  $\mathbb{R}^{m \times n}$

$$\begin{pmatrix} 1 & 0 & 2 & 4 \\ 7 & 6 & 8 & 4 \\ 0 & 0 & 2 & 1 \end{pmatrix} \quad 3 \times 4$$

2nd row

3rd col

- A row vector with n entries is also a matrix of size $1 \times n$.

  $(1 \quad 2 \quad 3)$ $\quad \mathbb{R}^{1 \times n}$

- A col vector with n entries is a matrix of size $n \times 1$.

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \qquad \frac{\mathbb{R}^{n \times 1}}{\mathbb{R}^{n}}$$

- $\mathbb{R}^{n \times n}$ is called square.

- Matrix – Vector Mul : $A \in \mathbb{R}^{m \times n}$

  $\vec{v} \in \mathbb{R}^{n}$

$$m \times n \quad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \quad n \times 1$$

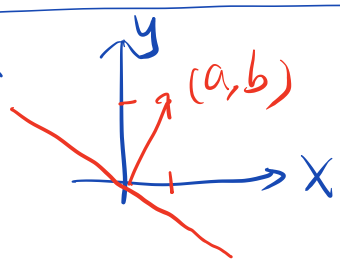
① $= a\begin{pmatrix}1\\5\end{pmatrix} + b\begin{pmatrix}2\\6\end{pmatrix} + c\begin{pmatrix}3\\7\end{pmatrix} + d\begin{pmatrix}4\\8\end{pmatrix}$

$$= \begin{pmatrix}a\\5a\end{pmatrix} + \begin{pmatrix}2b\\6b\end{pmatrix} + \begin{pmatrix}3c\\7c\end{pmatrix} + \begin{pmatrix}4d\\8d\end{pmatrix}$$

$$= \begin{pmatrix}a+2b+3c+4d\\5a+6b+7c+8d\end{pmatrix} \quad m\times 1$$

② $$\begin{pmatrix}\boxed{1 \quad 2 \quad 3 \quad 4}\\ \boxed{5 \quad 6 \quad 7 \quad 8}\end{pmatrix}\begin{pmatrix}a\\b\\c\\d\end{pmatrix} = \begin{pmatrix}*\\*\end{pmatrix}$$

2D:



$\boxed{ax+by=C}$

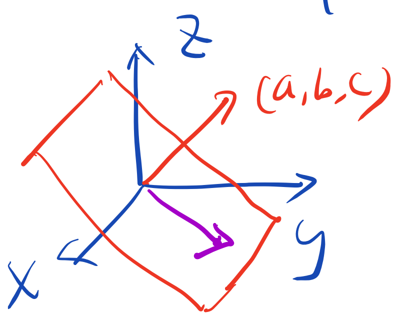
$$\Longleftrightarrow \quad y = -\frac{a}{b}x + \frac{c}{b}$$

Why is $ax+by=0$ a line Eqn?

$$\begin{pmatrix}a\\b\end{pmatrix}\cdot\begin{pmatrix}x\\y\end{pmatrix} = 0$$

$$\Longleftrightarrow \begin{pmatrix}a\\b\end{pmatrix} \perp \begin{pmatrix}x\\y\end{pmatrix}$$

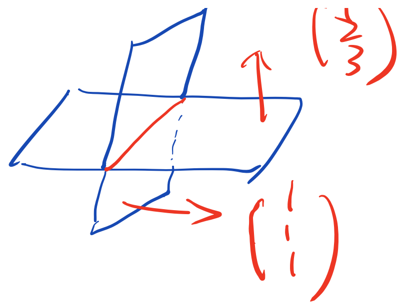
3D:



$ax+by+cz=0$

$$\begin{pmatrix}a\\b\\c\end{pmatrix}\cdot\begin{pmatrix}x\\y\\z\end{pmatrix} = 0$$

$$\begin{cases} x + 2y + 3z = 1 \\ x + y + z = 0 \end{cases}$$



$$\left.\begin{cases} x + 2y + 3z = 1 \\ x + y + z = 0 \end{cases}\right\} \text{line}$$
$$-x + y + z = -1$$

Matrix Form of Linear System

(System of linear equations)

$$\underbrace{\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix}}_{\substack{\text{Coef Matrix} \\ A}} \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\substack{\text{unknown vector} \\ \vec{x}}} = \underbrace{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}_{\substack{\text{RHS vector} \\ \vec{b}}}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 1 & 1 & 1 & 0 \\ -1 & 1 & 1 & -1 \end{array}\right)$$

Augmented Matrix

$$A\vec{x} = \vec{b}$$

$$\begin{pmatrix} x + 2y + 3z \\ x + y + z \\ -x + y + z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Ex: $\begin{cases} x + y + z = 0 \\ z - y = 1 \end{cases}$

$$\begin{cases} x + y + z = 0 \\ 0\cdot x - y + z = 1 \end{cases} \quad \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -1 & 1 & 1 \end{array}\right)$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- Augmented Matrix $[A \mid \vec{b}]$

- Row operations to Augmented Matrix

Type I Row Op: switch two rows

II : mul a row by a nonzero scala

III : mul a row by a scalar then add it to another row

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 1 & 1 & 1 & 0 \\ -1 & 1 & 1 & -1 \end{array}\right) \xRightarrow{r1 \Leftrightarrow r2}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 1 \\ -1 & 1 & 1 & -1 \end{array}\right) \xRightarrow{(-1)\cdot r3 \to r3}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 1 \\ 1 & -1 & -1 & 1 \end{array}\right) \xRightarrow{(-1)\cdot r1 + r2 \to r2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 1 & -1 & -1 & 1 \end{array}\right)$$

- The goal is to transform $[A \mid \vec{b}]$ into

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{array}\right]$$

This is called <u>Gaussian Elimination</u>

Ex: $\begin{cases} x + 2y + 3z = 1 \\ x + y + z = 0 \\ -x + y + z = -1 \end{cases}$

Sol: $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

Step 0: $\begin{array}{c} \text{leading one} \\ \text{pivot} \end{array}$ $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 1 & 1 & 1 & 0 \\ -1 & 1 & 1 & -1 \end{array}\right]$

Step 1: start with first nonzero col, do Type I ops to make (1,1)-entry to be nonzero.

Do Type II to make it one.

Step 2: Do Type III op to make all other entries zero in the col of the leading one.

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 1 & 1 & 1 & 0 \\ -1 & 1 & 1 & -1 \end{array}\right] \xrightarrow{(-1)\cdot r1+r2 \to r2}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -1 & -2 & -1 \\ -1 & 1 & 1 & -1 \end{array}\right] \xrightarrow{r1+r3 \to r3}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -1 & -2 & -1 \\ 0 & 3 & 4 & 0 \end{array}\right]$$

Step 3: do similar thing to the submatrix (ignoring 1st col)

$$\xrightarrow{(-1)\cdot r2 \to r2} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 3 & 4 & 0 \end{array}\right)$$

$$\xrightarrow{(-2)\cdot r2 + r1 \to r1} \left(\begin{array}{ccc|c} 1 & 0 & -1 & -1 \\ 0 & 1 & 2 & 1 \\ 0 & \cancel{3} & 4 & 0 \end{array}\right)$$

$$\xrightarrow{(-3)\cdot r2 + r3 \to r3} \left(\begin{array}{ccc|c} 1 & 0 & -1 & -1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -2 & -3 \end{array}\right)$$

$$\xrightarrow{(-\frac{1}{2})\cdot r3 \to r3} \left(\begin{array}{ccc|c} 1 & 0 & -1 & -1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 3/2 \end{array}\right)$$

$r3+r1 \to r1$

$$\Longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 3/2 \end{array}\right)$$

$(-2)\cdot r3+r2 \to r2$

$$\Longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3/2 \end{array}\right)$$

$$\begin{cases} 1\cdot x + 0\cdot y + 0\cdot z = \frac{1}{2} \\ 0\cdot x + 1\cdot y + 0\cdot z = -2 \\ 0\cdot x + 0\cdot y + 1\cdot z = 3/2 \end{cases}$$

Def Row Echelon Form (REF)

① zero rows are at bottom

② leading coef of a nonzero row
pivot
first nonzero entry in a row
is always strictly to the right of the pivot of the row above it.

Example: $\begin{pmatrix} 1 & a & b & c & 0 \\ 0 & 0 & 2 & d & f \\ 0 & 0 & 0 & 1 & g \end{pmatrix}$ REF

$\begin{pmatrix} 1 & a & b & c & 0 \\ 1 & 0 & 2 & d & f \\ 0 & 0 & 0 & 1 & g \end{pmatrix}$ ✗

$\begin{pmatrix} 1 & a & b & c & 0 \\ 0 & 0 & 2 & d & f \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & g \end{pmatrix}$ ✗

$\begin{pmatrix} 1 & a & b & c & 0 \\ 0 & 0 & 2 & d & f \\ 0 & 0 & 0 & 1 & g \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ ✓

Definition (Reduced Row Echelon Form)

A REF is called Reduced REF (RREF)

if
1. all pivots are ones. (leading one)
2. Columns containing leading ones has only one nonzero

Example: $\begin{pmatrix} 1 & a & b & c & 0 \\ 0 & 0 & 1 & d & f \\ 0 & 0 & 0 & 1 & g \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ is NOT RREF

$$\begin{pmatrix} 1 & a & 0 & b & 0 & 1 \\ 0 & 0 & 1 & c & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$ is RREF

$$\begin{pmatrix} 1 & a & 0 & b & 0 & 0 \\ 0 & 0 & 1 & c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$ is NOT REF