Review

- Matrix-Matrix Multiplication: for $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{n \times p}$, $AB \in \mathbb{R}^{m \times p}$ can be computed or interpreted/viewed as

① Each col in AB is a linear combination of cols in A.

$$\underset{2\times 4}{\begin{pmatrix} 1 & 2 & 3 & 4 \\ -1 & 1 & 0 & 2 \end{pmatrix}} \underset{4\times 3}{\begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 1 & -1 \\ 2 & 0 & 1 \end{pmatrix}} = \underset{2\times 3}{\begin{pmatrix} 7 & 5 & 2 \\ 2 & 1 & 1 \end{pmatrix}}$$

Ex: $\begin{pmatrix} 2 \\ 1 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} + 0 \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} + (-1) \cdot \begin{pmatrix} 3 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 4 \\ 2 \end{pmatrix}$

$= \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -3 \\ 0 \end{pmatrix} + \begin{pmatrix} 4 \\ 2 \end{pmatrix}$

② Each row in AB is a linear combination of rows in B.

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ -1 & 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 1 & -1 \\ 2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 7 & 5 & 2 \\ 2 & 1 & 1 \end{pmatrix}$$

$(2 \ 1 \ 1) = (-1) \cdot (1 \ 0 \ 1) + 1 \cdot (-1 \ 1 \ 0) + 0 \cdot (0 \ 1 \ -1)$
$+ 2 \cdot (2 \ 0 \ 1)$

$= (-1 \ 0 \ -1) + (-1 \ 1 \ 0)$

$+ (0 \ 0 \ 0) + (4 \ 0 \ 2)$

- Identity Matrix : $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

  $AI = A$, $IA = A$.

- Inverse Matrix : $A \in \mathbb{R}^{n\times n}$, $AA^{-1} = I$

  $A^{-1}A = I$

  Ex: $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

- Assume $A \in \mathbb{R}^{n\times n}$ is invertible, then

  ① A^{-1} is unique

  ② $(A^{-1})^{-1} = A$ ③ If $AB = I$, then $BA = I$.

- Elementary Matrices : generated by one row operation on the identity matrix

E is always invertible. E^{-1} is also an elementary matrix, generated by the inverse operation.

Type I $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, $E^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

Type II $E = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$, $E^{-1} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}$

Type III $E = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $E^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$

① EA is equivalent to row op on A

② Let $E_1, E_2, E_3, \cdots, E_m$ be the elementary matrices corresponding to row operations turing an invertible A to I, then $E_m \cdots E_2 E_1 A = I$

$E_m \cdots E_1 \boxed{A} = \boxed{RREF}$

③ $A^{-1} = E_m E_{m-1} \cdots E_2 E_1$

④ $A = E_1^{-1} E_2^{-1} \cdots E_m^{-1}$

⑤ Any invertible matrix is a product of some elementary matrices.

$A^{-1}B^{-1} \neq B^{-1}A^{-1}$

Fact: assume $A, B \in \mathbb{R}^{n\times n}$ are invertible

then $(AB)\cdot(B^{-1}A^{-1}) = I$

$ABB^{-1}A^{-1}$
$= A(BB^{-1})A^{-1}$
$= A \underline{IA^{-1}} = AA^{-1}$
$= I$

So ① $(AB)^{-1} = B^{-1}A^{-1}$

② $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

Chapter 3 Vector Spaces

- If a vector $\vec{v}$ is a linear combination of some vectors, then we say $\vec{v}$ is spanned by these vectors.

Example: $A\vec{x}=\vec{b}$

$$\begin{pmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 2 \\ 8 \\ 10 \end{pmatrix}$$ has one sol $\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}=\begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}=\begin{pmatrix} 2 \\ 8 \\ 10 \end{pmatrix}$$

Remark: still a span if x_0, y_0, z_0 can be 0.

$$\Rightarrow \begin{pmatrix} 2 \\ 8 \\ 10 \end{pmatrix} = x_0\begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix} + y_0\begin{pmatrix} 4 \\ 9 \\ -3 \end{pmatrix} + z_0\begin{pmatrix} -2 \\ -3 \\ 7 \end{pmatrix}$$

$\Rightarrow \vec{b}$ is spanned by cols of A.

- S is a set of vectors, Span(S) denotes the set of all possible linear combination of vectors in S.

Example: ① $\text{Span}\left\{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right\}=\left\{a\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \forall a\in\mathbb{R}\right\}$ is a line.

② $\text{Span}\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}=\left\{a\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}+b\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \forall a,b\in\mathbb{R}\right\}$

③ $\text{Span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}=\mathbb{R}^3$

- Definition (Abstract Vectors)

A Vector Space over real numbers is a set S and

① addition is defined for any two elements in the set

② scalar multiplication is defined for a scalar in $\mathbb{R}$ and any element in the set.

③ the set is closed under these two operations.

meaning that

1) the sum of two element is still in the set S

2) scalar multiplication is still in the set S

④ elements in this set is called (abstract) vectors.

Example: the following are all (abstract) vector spaces

$\mathbb{R} = \{\text{all real numbers}\}$

elementary vectors

$\mathbb{R}^2 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix}, \forall x, y \in \mathbb{R} \right\}$

$\mathbb{R}^3 = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \forall a, b, c \in \mathbb{R} \right\}$

$\mathbb{R}^4 = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}, \forall a, b, c, d \in \mathbb{R} \right\}$

$\mathbb{R}^{1\times 3} = \left\{ \begin{bmatrix} x & y & z \end{bmatrix} : \forall x, y, z \in \mathbb{R} \right\}$

$\mathbb{R}^{2\times 2} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \forall a, b, c, d \in \mathbb{R} \right\}$

$\mathbb{R}^{m\times n}$

$P_2(x) = \{ ax^2 + bx + c : \forall a, b, c \in \mathbb{R} \}$

Ex: Is $\operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$ a Vector Space? Yes

Sol: $\forall \vec{u}, \vec{v} \in \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$, $\forall a \in \mathbb{R}$

$\vec{u} \in \text{Span}\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}\right\} \Rightarrow \vec{u} = s\begin{bmatrix}1\\2\\3\end{bmatrix},\ s \in \mathbb{R}.$

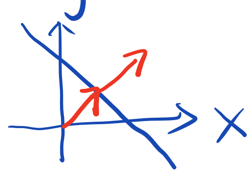
$a\vec{u} = a \cdot s\begin{bmatrix}1\\2\\3\end{bmatrix} = (as)\begin{bmatrix}1\\2\\3\end{bmatrix} \Rightarrow a\vec{u} \in \text{Span}\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}\right\}$

$\vec{v} \in \text{Span}\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}\right\} \Rightarrow \vec{v} = t\begin{bmatrix}1\\2\\3\end{bmatrix},\ t \in \mathbb{R}.$

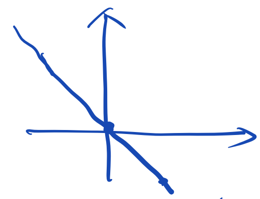
$\vec{u} + \vec{v} = (s+t)\begin{bmatrix}1\\2\\3\end{bmatrix} \in \text{Span}\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}\right\}.$

Ex: Is $\text{Span}\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$ a vector space? Yes

Ex:



$y = -x+1$



$y = -x$

- There is always a zero vector $\vec{0}$ s.t. $\vec{v} + \vec{0} = \vec{v}$

  1) For $\mathbb{R}^n$, $\vec{0} = \begin{pmatrix}0\\0\\\vdots\\0\end{pmatrix}$

  2) For $\mathbb{R}^{m\times n}$, $\vec{0}$ is the zero matrix of size $m\times n$.

  3) For $P_2(\mathbb{R})$, $\vec{0}$ is the zero polynomial $p(x) = 0$.

- Theorem: Let V be an abstract vector space.

  1) $0 \cdot \vec{v} = \vec{0}$, $\forall \vec{v} \in V$

  2) Closedness $\Rightarrow \vec{0} = 0 \cdot \vec{v} \in V$.