- Dot (Inner) Product : $\vec{x}, \vec{y} \in \mathbb{R}^n$

$$\langle \vec{x}, \vec{y} \rangle = \vec{y}^T \vec{x} = (y_1\ y_2 \cdots y_n) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$= \sum_{i=1}^{n} x_i y_i$$

$$= x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

- Length : $\|\vec{x}\| = \sqrt{\langle \vec{x}, \vec{x} \rangle} = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$

- Orthogonality : we say $\vec{x} \perp \vec{y}$ if $\langle \vec{x}, \vec{y} \rangle = 0$

- Properties of Dot Product: $\forall \vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^n$, $a, b \in \mathbb{R}$

① Linearity: $\langle a\vec{x} + b\vec{y}, \vec{z} \rangle = a\langle \vec{x}, \vec{z} \rangle + b\langle \vec{y}, \vec{z} \rangle$

$\langle \vec{z}, a\vec{x} + b\vec{y} \rangle = a\langle \vec{z}, \vec{x} \rangle + b\langle \vec{z}, \vec{y} \rangle$

② Symmetry : $\langle \vec{x}, \vec{y} \rangle = \langle \vec{y}, \vec{x} \rangle$

③ $\langle \vec{x}, \vec{x} \rangle \geq 0$

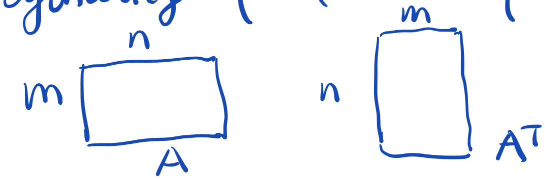
and

$\langle \vec{x}, \vec{x} \rangle = 0 \iff \vec{x} = \vec{0}$

Any operation on two vectors satisfying these 3 properties is called an Inner Product.

Dot product is the simplest inner product.

Orthogonality of four subspaces of $A \in \mathbb{R}^{m \times n}$:



① Column Space of A : $Col(A) \subseteq \mathbb{R}^m$

② Row Space of A : $Col(A^T) \subseteq \mathbb{R}^n$

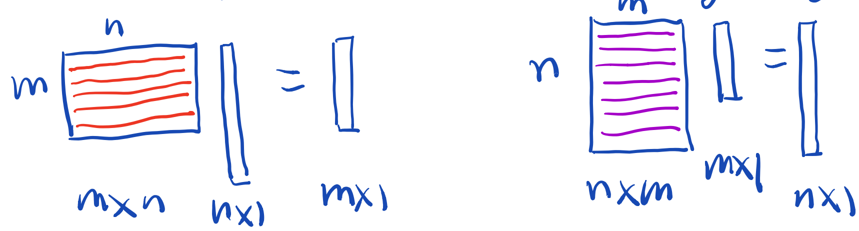
③ Null Space of A : $Null(A) \subseteq \mathbb{R}^n$

④ Left Null Space of A : $Null(A^T) \subseteq \mathbb{R}^m$

$$Null(A) = \{\vec{x} \in \mathbb{R}^n : \boxed{A\vec{x} = \vec{0}}\}$$

$$Null(A^T) = \{\vec{y} \in \mathbb{R}^m : A^T\vec{y} = \vec{0}\}$$

$A\vec{x} = \vec{0}$ $\qquad\qquad$ $A^T\vec{y} = \vec{0}$



If $A\vec{x} = \vec{0}$, then each row of A is $\perp$ to $\vec{x}$

$\Rightarrow \boxed{Col(A^T)} \perp Null(A)$

If $A^T\vec{y} = \vec{0}$, then each row of A^T is $\perp$ to $\vec{y}$

$\Rightarrow$ each col of A is $\perp$ to $\vec{y}$

$\Rightarrow Col(A) \perp Null(A^T)$

Example: Find four subspaces of

$$A = \begin{pmatrix} 1 & 3 & 2 & 2 \\ 1 & 2 & 3 & 1 \\ 0 & 1 & -1 & 1 \end{pmatrix}, \quad A^T = \begin{pmatrix} 1 & 1 & 0 \\ 3 & 2 & 1 \\ 2 & 3 & -1 \\ 2 & 1 & 1 \end{pmatrix}$$

3x4

$$\mathrm{RREF}(A|\vec{0})\quad \left(\begin{array}{cccc|c} 1 & 3 & 2 & 2 & 0 \\ 1 & 2 & 3 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} 1 & 3 & 2 & 2 & 0 \\ 0 & -1 & 1 & -1 & 0 \\ 0 & 1 & -1 & 1 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} 1 & 3 & 2 & 2 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\to \begin{array}{c} \left(\begin{array}{cccc|c} 1 & 0 & 5 & -1 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \\ \begin{array}{cccc} x & y & z & w \end{array} \end{array}$$

$$\begin{cases} z = s \\ w = t \end{cases} \Rightarrow \begin{cases} x = -5s+t \\ y = s-t \end{cases} \Rightarrow \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -5s+t \\ s-t \\ s \\ t \end{pmatrix}$$

$$= \begin{pmatrix} -5s \\ s \\ s \\ 0 \end{pmatrix} + \begin{pmatrix} t \\ -t \\ 0 \\ t \end{pmatrix} = s\begin{pmatrix} -5 \\ 1 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix},\ \forall s,t \in \mathbb{R}$$

$$\Rightarrow \mathrm{Null}(A) = \mathrm{Span}\left\{ \begin{bmatrix} -5 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\} \subseteq \mathbb{R}^4$$

$$\mathrm{Col}(A) = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \right\} \subseteq \mathbb{R}^3$$

$RREF(A^T|\vec{0})$

$$\left(\boxed{\begin{matrix} 1 & 1 & 0 \\ 3 & 2 & 1 \\ 2 & 3 & -1 \\ 2 & 1 & 1 \end{matrix}} \;\middle|\; \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \end{matrix}\right)$$

$$\rightarrow \left(\begin{matrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{matrix} \;\middle|\; \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \end{matrix}\right)$$

$$\rightarrow \left(\begin{matrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \;\middle|\; \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \end{matrix}\right)$$

$$\rightarrow \begin{array}{c} \left(\begin{matrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \;\middle|\; \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \end{matrix}\right) \\ \begin{matrix} x & \;\; y & \;\; z \end{matrix} \end{array}$$

$$z = t \Rightarrow \begin{cases} x = -t \\ y = t \end{cases} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -t \\ t \\ t \end{pmatrix} = t \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \quad \forall t \in \mathbb{R}$$

$$\Rightarrow Null(A^T) = Span\left\{\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}\right\} \subseteq \mathbb{R}^3$$

$$Col(A^T) = Span\left\{\begin{bmatrix} 1 \\ 3 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}\right\} \subseteq \mathbb{R}^4$$

$rank(A) = rank(A^T) = 2$

$Col(A^T) \perp Null(A)$

$$\mathrm{Null}(A) = \mathrm{Span}\left\{ \begin{bmatrix} -5 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\} \subseteq \mathbb{R}^4$$

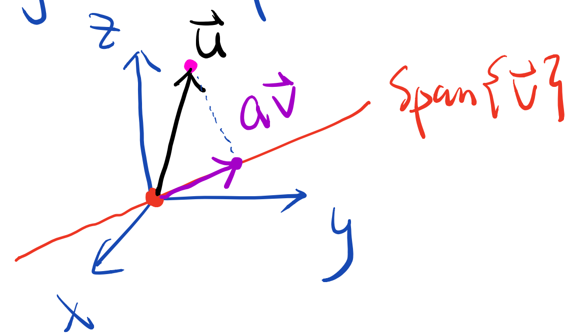
$A : 3 \times 4$

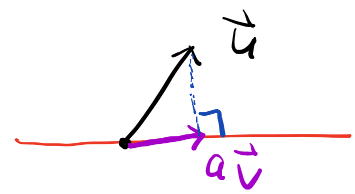
$A^T : 4 \times 3$

$$\mathrm{Col}(A) = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \right\} \subseteq \mathbb{R}^3$$

$\mathrm{Col}(A) \perp \mathrm{Null}(A^T)$

Projection of a vector onto another one:





So we look for a vector $a\vec{v}$ satisfying

$$(\vec{u} - a\vec{v}) \perp \vec{v}$$

$$\Leftrightarrow \quad \langle \vec{u} - a\vec{v}, \vec{v} \rangle = 0$$

$$\Leftrightarrow \quad \langle \vec{u}, \vec{v} \rangle - a \langle \vec{v}, \vec{v} \rangle = 0$$

$$\Leftrightarrow \quad a \langle \vec{v}, \vec{v} \rangle = \langle \vec{u}, \vec{v} \rangle$$

$$\Leftrightarrow \quad a = \frac{\langle \vec{u}, \vec{v} \rangle}{\langle \vec{v}, \vec{v} \rangle} = \frac{\vec{v}^T \vec{u}}{\vec{v}^T \vec{v}} = \frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}}$$

$$= \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2}$$

For $\vec{u}, \vec{v} \in \mathbb{R}^n$, the projection of $\vec{u}$ onto $\vec{v}$

is $$P_{\vec{v}}(\vec{u}) = \frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\vec{v}$$

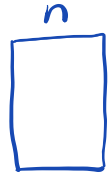
Ex: Projection of $\vec{u} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ onto $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$P_{\vec{v}}(\vec{u}) = \frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\vec{v} = \frac{6}{(1+1+1)}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}.$$

Motivation for Projection

$$A\vec{x} = \vec{b} \qquad A \in \mathbb{R}^{m\times n}$$

n

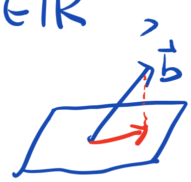
m  $m > n$

$A\vec{x} = \vec{b}$ has at least one sol

$\iff \vec{b} \in Col(A) \subseteq \mathbb{R}^m$

$dim(Col(A)) \leq n < m$

$\vec{b} \in \mathbb{R}^m$, $Col(A) \neq \mathbb{R}^m$



Col(A)