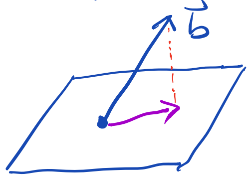
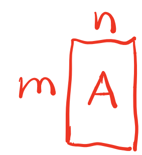
- Given $\vec{v}_1, \cdots, \vec{v}_k \in \mathbb{R}^m$, want to find the projection of $\vec{b}$ onto $\text{Span}\{\vec{v}_1, \cdots, \vec{v}_k\}$



Step I : Find a basis for $\text{Span}\{\vec{v}_1, \cdots, \vec{v}_k\}$

Suppose the basis is $\{\vec{v}_1, \cdots, \vec{v}_n\}$



$n \leq m$

Step II : Form a matrix $A \in \mathbb{R}^{m \times n}$ by putting cols $\vec{v}_1, \cdots, \vec{v}_n$ together.

Both the shortest distance and orthogonality imply $\underbrace{A^T A}\hat{x} = A^T \vec{b}$, which implies the projection formula is

$$A(A^TA)^{-1}A^T\vec{b}$$

Example: Find projection of $\vec{b} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ onto $\text{Span}\left\{ \begin{pmatrix} 1 \\ -2 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 1 \\ -2 \end{pmatrix} \right\}$

Sol : Step I : Find a basis for the Span.

$$a\underbrace{\begin{pmatrix} 1 \\ -2 \\ -1 \\ -1 \end{pmatrix}} + b\underbrace{\begin{pmatrix} 0 \\ 3 \\ 0 \\ 3 \end{pmatrix}} + c\underbrace{\begin{pmatrix} 2 \\ 1 \\ -1 \\ 3 \end{pmatrix}} + d\begin{pmatrix} -1 \\ -1 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 2 & -1 & 0 \\ -2 & 3 & 1 & -1 & 0 \\ -1 & 0 & -1 & 1 & 0 \\ -1 & 3 & 3 & -2 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$\Rightarrow$ A basis is $\left\{ \begin{pmatrix} 1 \\ -2 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ -1 \\ 3 \end{pmatrix} \right\}$

Step II: $A = \begin{pmatrix} 1 & 0 & 2 \\ -2 & 3 & 1 \\ -1 & 0 & -1 \\ -1 & 3 & 3 \end{pmatrix}$

$$A^T A \hat{x} = A^T \vec{b}$$

$$\begin{pmatrix} 1 & -2 & -1 & -1 \\ 0 & 3 & 0 & 3 \\ 2 & 1 & -1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ -2 & 3 & 1 \\ -1 & 0 & -1 \\ -1 & 3 & 3 \end{pmatrix} \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{pmatrix} = \begin{pmatrix} 1 & -2 & -1 & -1 \\ 0 & 3 & 0 & 3 \\ 2 & 1 & -1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

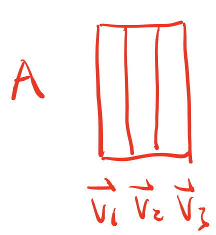
$$\begin{pmatrix} 7 & -9 & -2 \\ -9 & 18 & 12 \\ -2 & 12 & 15 \end{pmatrix} \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

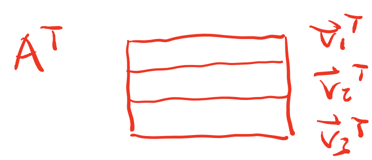
After solving $\hat{x}$, we still need to compute

$$A\hat{x} = \begin{pmatrix} 1 & 0 & 2 \\ -2 & 3 & 1 \\ -1 & 0 & -1 \\ -1 & 3 & 3 \end{pmatrix} \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{pmatrix}$$

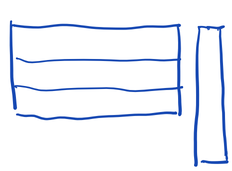
- If cols of A are independent, then $A^T A$ is invertible.

The projection matrix is $A(A^T A)^{-1} A^T$

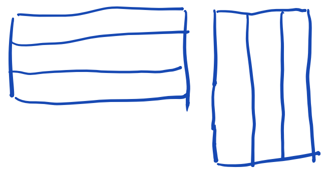




$A^T \vec{b} =$



$= \begin{pmatrix} \vec{v}_1^T \vec{b} \\ \vec{v}_2^T \vec{b} \\ \vec{v}_3^T \vec{b} \end{pmatrix}$

$$A^TA = $$  $$= \begin{pmatrix} \vec{v}_1^T\vec{v}_1 & \vec{v}_1^T\vec{v}_2 & \vec{v}_1^T\vec{v}_3 \\ \vec{v}_2^T\vec{v}_1 & \vec{v}_2^T\vec{v}_2 & \vec{v}_2^T\vec{v}_3 \\ \vec{v}_3^T\vec{v}_1 & \vec{v}_3^T\vec{v}_2 & \vec{v}_3^T\vec{v}_3 \end{pmatrix}$$

(i,j)-entry is $\vec{v}_i^T\vec{v}_j$

- $\{\vec{v}_1, \cdots, \vec{v}_j\}$ is orthogonal if $\vec{v}_i^T\vec{v}_j = 0$, $i \neq j$

  $\{\vec{v}_1, \cdots, \vec{v}_j\}$ is orthonormal if $\vec{v}_i^T\vec{v}_j = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$

  $\|\vec{v}\|^2 = \langle \vec{v}, \vec{v} \rangle = \vec{v}^T\vec{v}$

  If orthogonal, $A^TA = \begin{pmatrix} \|\vec{v}_1\|^2 & 0 & 0 \\ 0 & \|\vec{v}_2\|^2 & 0 \\ 0 & 0 & \|\vec{v}_3\|^2 \end{pmatrix}$

$$(A^TA)^{-1} = \begin{pmatrix} 1/\|\vec{v}_1\|^2 & 0 & 0 \\ 0 & 1/\|\vec{v}_2\|^2 & 0 \\ 0 & 0 & 1/\|\vec{v}_3\|^2 \end{pmatrix}$$

  If orthonormal, $A^TA = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

- The projection of $\vec{b}$ is $A(A^TA)^{-1}A^T\vec{b}$, which can also be written out as

① $\dfrac{\langle \vec{v}_1, \vec{b} \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 + \dfrac{\langle \vec{v}_2, \vec{b} \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 + \dfrac{\langle \vec{v}_3, \vec{b} \rangle}{\|\vec{v}_3\|^2}\vec{v}_3$ if orthogonal.

② $\langle \vec{v}_1, \vec{b} \rangle \vec{v}_1 + \langle \vec{v}_2, \vec{b} \rangle \vec{v}_2 + \langle \vec{v}_3, \vec{b} \rangle \vec{v}_3$ if orthonormal.

Remark : $\frac{\langle \vec{v}_i, \vec{b} \rangle}{\|\vec{v}_i\|^2} \vec{v}_i$ is the projection of $\vec{b}$ onto $\text{Span}\{\vec{v}_i\}$

Gram-Schmidt Process :

Given a basis $\{\vec{w}_1, \cdots, \vec{w}_n\}$ of some subspace,

want an orthonormal / orthogonal basis.

Version I :

$$\left.\begin{aligned} v_1 &= w_1. \\ v_2 &= w_2 - \frac{\langle w_2, v_1 \rangle}{\|v_1\|^2} v_1 \\ v_3 &= w_3 - \frac{\langle w_3, v_1 \rangle}{\|v_1\|^2} v_1 - \frac{\langle w_3, v_2 \rangle}{\|v_2\|^2} v_2 \end{aligned}\right\} \vec{v}_1, \vec{v}_2, \vec{v}_3 \text{ are orthogonal}$$

$$u_1 = \frac{v_1}{\|v_1\|}$$

$$u_2 = \frac{v_2}{\|v_2\|}$$

$$u_3 = \frac{v_3}{\|v_3\|}.$$

Version II :

① $\begin{cases} \vec{v}_1 = \vec{w}_1 \\ \vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} \end{cases}$

② $\begin{cases} \vec{v}_2 = \vec{w}_2 - \langle \vec{w}_2, \vec{u}_1 \rangle \vec{u}_1 \\ \vec{u}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|} \end{cases}$

③ $\begin{cases} \vec{v}_3 = \vec{w}_3 - \langle \vec{w}_3, \vec{u}_1 \rangle \vec{u}_1 - \langle \vec{w}_3, \vec{u}_2 \rangle \vec{u}_2 \\ \vec{u}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|} \end{cases}$

Chapter 5 Derterminants of Square Matrices

det (A) or |A| means determinant

The following are equivalent for $A \in \mathbb{R}^{n\times n}$

dim Theorem: Nullity + Rank = # of cols

① A^{-1} exists

② rank (A) = n.

③ Nullity (A) = 0 ($\Leftrightarrow$ Rank (A) = n)

④ $Ax = 0$ has only zero sol.

⑤ $Ax = b$ has a unique sol $A^{-1}b$.

⑥ row space of A has dim n

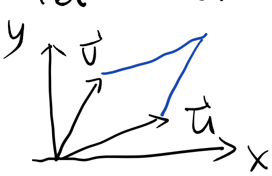
⑦ col ______ ______

⑧ $A = E_1 \cdots E_n$ a product of elemetary matrices

⑨ $\det(A) \neq 0$

$A \in \mathbb{R}^{n\times n}$, $\det(A) = |A|$ is a scalar

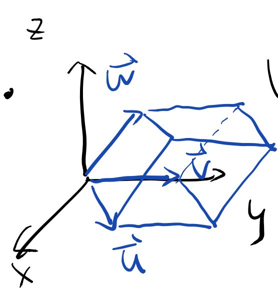
- Det in Calculus: Area of parallelegram generated by $\vec{u} = \begin{pmatrix} a \\ b \end{pmatrix}$, $\vec{v} = \begin{pmatrix} c \\ d \end{pmatrix}$



is $\left|\det\begin{pmatrix} a & c \\ b & d \end{pmatrix}\right| = |ad - bc|$

$= \left|\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right| = |ad - bc|$

- Volume of parallelepiped generated by $\vec{u}, \vec{v}, \vec{w}$
$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \begin{pmatrix} d \\ e \\ f \end{pmatrix} \begin{pmatrix} g \\ h \\ i \end{pmatrix}$



$= |\vec{u} \times \vec{v} \cdot \vec{w}|$

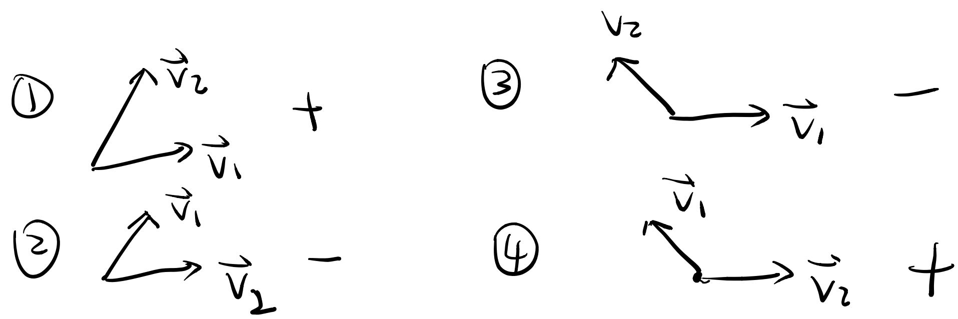
$$= \left| \det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \right| \rightarrow \text{the absolute value of det.}$$

$$\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = +aei + bfg + cdh - ceg - fha - bdi$$

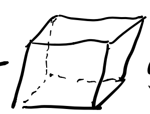
- Geometric Interpretation of $\det(A)$ for $A \in \mathbb{R}^{2\times 2}$:

  1) $|\det(A)|$ is the area of  generated by two cols $\vec{v}_1, \vec{v}_2$ of A

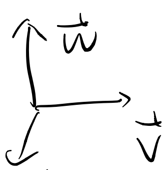
  2) $\det(A)$ has a sign: $+$ or $-$



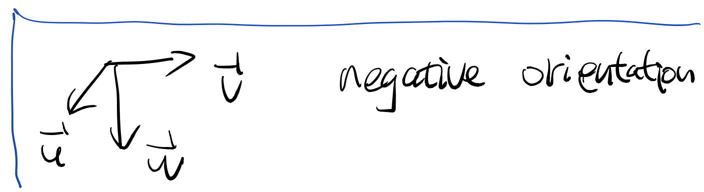
- Geometric Interpretation of $\det(A)$ for $A \in \mathbb{R}^{3\times 3}$:

  1) $|\det(A)|$ is the vol of  generated by its cols $\vec{u}, \vec{v}, \vec{w}$

  2) The sign means 3D orientation of $\vec{u}, \vec{v}, \vec{w}$ by Right Hand Rule.



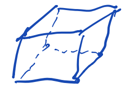
positive orientation



- Want to define $\det(A)$ for $A \in \mathbb{R}^{n\times n}$ s.t.

1) $|\det(A)|$ is $\begin{cases} \text{area of } \square \text{ in 2D} \\ \text{volume of } \text{in 3D} \end{cases}$

area of  in 2D

volume of  in 3D

2) $\det(A)$ is a linear function w.r.t. each col of A.

3) Switching any two cols will change sign of $\det(A)$

4) $\det(I) = 1$ $\quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ $\quad \begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

It turns out that such a function is unique if satisfying all these. We define the n-dimensional "volume" of the n-dimensional "parallelipiped" as $|\det(A)|$

Def For $A \in \mathbb{R}^{n\times n}$, A_{ij} denotes its entry in $\begin{cases} \text{row } i \\ \text{col } j. \end{cases}$

The cofactor matrix of A_{ij} is the $(n-1)\times(n-1)$ matrix obtained by deleting i-th row and j-th col in A.

Ex: $A = \begin{pmatrix} 1 & -1 & 2 & 3 \\ 3 & 4 & \boxed{1} & 2 \\ 2 & -5 & -3 & 8 \\ -6 & 2 & -4 & 1 \end{pmatrix}$ $\qquad A_{23} = 1$

The cofactor matrix of A_{23} is $\begin{pmatrix} 1 & -1 & 3 \\ 2 & -5 & 8 \\ -6 & 2 & 1 \end{pmatrix}$

Def For $A \in \mathbb{R}^{n\times n}$, det can be defined recursively

as $\det(A) = \sum_{j=1}^{n} (-1)^{i+j} A_{ij} \det(\tilde{A}_{ij})$ $\quad$ Cofactor expansion along i-th row

$\tilde{A}_{ij}$ denotes the cofactor matrix of entry A_{ij}

or $\det(A) = \sum_{i=1}^{n} (-1)^{i+j} A_{ij} \det(\tilde{A}_{ij})$ cofactor expansion along j-th col

$$\det\begin{pmatrix} 1 & -1 & 2 & 3 \\ 3 & 4 & 1 & 2 \\ 2 & -5 & -3 & 8 \\ -6 & 2 & -4 & 1 \end{pmatrix} \overset{\text{(2nd row)}}{=} (-1)^{2+1} \cdot \boxed{3} \cdot \begin{vmatrix} -1 & 2 & 3 \\ -5 & -3 & 8 \\ 2 & -4 & 1 \end{vmatrix}$$

$$+ (-1)^{2+2} \cdot \boxed{4} \cdot \begin{vmatrix} 1 & 2 & 3 \\ 2 & -3 & 8 \\ -6 & -4 & 1 \end{vmatrix}$$

$$+ (-1)^{2+3} \cdot \boxed{1} \cdot \begin{vmatrix} 1 & -1 & 3 \\ 2 & -5 & 8 \\ -6 & 2 & 1 \end{vmatrix}$$

$$+ (-1)^{2+4} \cdot \boxed{2} \cdot \begin{vmatrix} 1 & -1 & 2 \\ 2 & -5 & -3 \\ -6 & 2 & -4 \end{vmatrix}$$

$(-1)^{i+j}$

Remark: ① For $A \in \mathbb{R}^{4\times4}$, compute it by four 3×3 dets.

⑤ For $A \in \mathbb{R}^{5\times5}$, compute it by five 4×4 dets.

Facts: ① Type 1 row/col ops changes det by (-1).

② Type 2 row/col ops: multiply a row/col by k will multiply k to det.

③ Type 3 row/col ops: no chan es to det.

④ $\det(I) = 1$.

⑤ zero row/col $\Rightarrow$ det $= 0$

⑥ same rows/cols $\Rightarrow$ det $= 0$

Ex: $\begin{vmatrix} 2 & 4 & 6 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} = 2\begin{vmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} = 2\begin{vmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 4 & 5 & 6 \end{vmatrix} = 0$

Ex: $\begin{vmatrix} 2 & 4 & 6 \\ 2 & 2 & 2 \\ 4 & 4 & 4 \end{vmatrix} = 8\begin{vmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{vmatrix}$

Example: $\begin{vmatrix} 1 & 3 & -3 \\ -3 & -5 & 2 \\ -4 & 4 & -6 \end{vmatrix} \overset{(3\cdot r_1 + r_2 \to r_2)}{=} \begin{vmatrix} 1 & 3 & -3 \\ 0 & 4 & -7 \\ -4 & 4 & -6 \end{vmatrix}$

$(4r_1 + r_3 \to r_3) = \begin{vmatrix} 1 & 3 & -3 \\ 0 & 4 & -7 \\ 0 & 16 & -18 \end{vmatrix}$

$= 2\begin{vmatrix} 1 & 3 & -3 \\ 0 & 4 & -7 \\ 0 & 8 & -9 \end{vmatrix}$

$= 2\Big[(-1)^{1+1} \cdot 1 \cdot \begin{vmatrix} 4 & -7 \\ 8 & -9 \end{vmatrix}$

$+ (-1)^{2+1} \cdot 0 \cdot \begin{vmatrix} 3 & -3 \\ 8 & -9 \end{vmatrix}$

$+ (-1)^{3+1} \cdot 0 \cdot \begin{vmatrix} 3 & -3 \\ 4 & -7 \end{vmatrix} \Big]$

$= 2 \cdot 1 \cdot \begin{vmatrix} 4 & -7 \\ 8 & -9 \end{vmatrix} = 2 \cdot (-36 + 56) = 40.$

$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

Example: $\begin{vmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 3 & -3 \\ -2 & -3 & -5 & 2 \\ 4 & -4 & 4 & -6 \end{vmatrix}$

$(C1 \cdot (-\frac{1}{2}) + C4 \to C4)$

$$= \begin{vmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 3 & -3 \\ -2 & -3 & -5 & 3 \\ 4 & -4 & 4 & -8 \end{vmatrix}$$

$$= 4 \begin{vmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 3 & -3 \\ -2 & -3 & -5 & 3 \\ 1 & -1 & 1 & -2 \end{vmatrix}$$

$$= 8 \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & -3 \\ -2 & -3 & -5 & 3 \\ 1 & -1 & 1 & -2 \end{vmatrix}$$

(cofactor expansion along first row)

$$= 8 \cdot (-1)^{1+1} \cdot 1 \cdot \begin{vmatrix} 1 & 3 & -3 \\ -3 & -5 & 3 \\ -1 & 1 & -2 \end{vmatrix}$$

$$= 8 \begin{vmatrix} 1 & 3 & -3 \\ -3 & -5 & 3 \\ -1 & 1 & -2 \end{vmatrix}$$

$$= 8 \begin{vmatrix} 1 & 3 & -3 \\ 0 & 4 & -6 \\ 0 & 4 & -5 \end{vmatrix}$$

$$= 8 \cdot (-1)^{1+1} \cdot 1 \cdot \begin{vmatrix} 4 & -6 \\ 4 & -5 \end{vmatrix}$$

$$= 8 \cdot (-20 + 24) = 32.$$

Some formulae about Det :

① $\det(A^T) = \det(A)$

② $\det(AB) = \det(A)\det(B)$

③ $\det(I) = \det(AA^{-1}) = \det(A)\det(A^{-1})$

$$\Rightarrow \det(A^{-1}) = \frac{1}{\det(A)}$$

$$\begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -1.$$

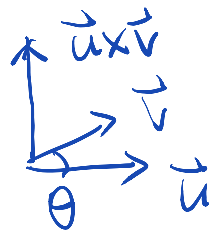
$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 2\begin{vmatrix} 1 & & \\ & 1 & \\ & & 1 \end{vmatrix} = 2.$$

- Cross Product of $\vec{u} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, $\vec{v} = \begin{pmatrix} d \\ e \\ f \end{pmatrix}$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a & b & c \\ d & e & f \end{vmatrix} \qquad \vec{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \vec{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \vec{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= (-1)^{1+1}\cdot \vec{i}\cdot \begin{vmatrix} b & c \\ e & f \end{vmatrix} + (-1)^{1+2}\cdot \vec{j} \begin{vmatrix} a & c \\ d & f \end{vmatrix} + (-1)^{1+3}\cdot \vec{k}\cdot \begin{vmatrix} a & b \\ d & e \end{vmatrix}$$

$$= (bf - ec)\vec{i} + [-(af - cd)]\vec{j} + (ae - bd)\vec{k}$$



$\|\vec{u}\times\vec{v}\| = \|\vec{u}\|\cdot\|\vec{v}\|\cdot\sin\theta$

is also area of ▱

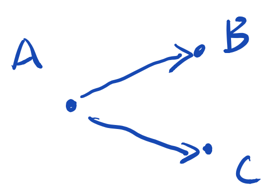
$\vec{u}\cdot\vec{v} = \|\vec{u}\|\cdot\|\vec{v}\|\cdot\cos\theta$

Ex: A $(1, 1, 1)$

B $(2, 1, 1)$

C $(1, 2, 3)$

Find area of ΔABC.



$\vec{u} = \overrightarrow{AB} = B - A = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$\vec{v} = \overrightarrow{AC} = C - A = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$

Area of $\Delta = \frac{1}{2}\|\vec{u}\times\vec{v}\|$