- Given $A \in \mathbb{R}^{n\times n}$, A_{ij} denotes its (i,j)-th entry, $\tilde{A}_{ij}$ denotes the cofactor matrix of A_{ij}, obtained by deleting the row and column containing A_{ij}

Ex: $A = \begin{pmatrix} 1 & -1 & 2 & 3 \\ 3 & 4 & \boxed{1} & 2 \\ 2 & -5 & -3 & 8 \\ -6 & 2 & -4 & 1 \end{pmatrix}$ $\qquad A_{23} = 1$

The cofactor matrix of A_{23} is $\begin{pmatrix} 1 & -1 & 3 \\ 2 & -5 & 8 \\ -6 & 2 & 1 \end{pmatrix}$

- Definition of det by cofactor expansion

① $A \in \mathbb{R}^{1\times 1}$ $\qquad A = [a]$, $\det(A) = a$.

② $A \in \mathbb{R}^{2\times 2}$ $\qquad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\begin{aligned} \det(A) = |A| &= (-1)^{1+1} \cdot a \cdot d \\ &\quad + (-1)^{1+2} \cdot c \cdot b \\ &= ad - bc \end{aligned}$$

③ $A \in \mathbb{R}^{3\times 3}$ $\qquad A = \begin{bmatrix} a & b & c \\ d & e & f \\ i & g & h \end{bmatrix}$

$$\begin{aligned} \det(A) = |A| &= (-1)^{1+2} \cdot b \cdot \begin{vmatrix} d & f \\ i & h \end{vmatrix} \\ &\quad + (-1)^{2+2} \cdot e \cdot \begin{vmatrix} a & c \\ i & h \end{vmatrix} \\ &\quad + (-1)^{3+2} \cdot g \cdot \begin{vmatrix} a & c \\ d & f \end{vmatrix} \end{aligned}$$

$$= -b\cdot(dh - if)$$
$$+ e(ah - ic)$$
$$- g(af - cd)$$

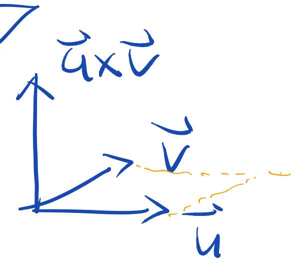
Example: The cross product of $\vec{u} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, $\vec{v} = \begin{pmatrix} d \\ e \\ f \end{pmatrix}$

is $\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a & b & c \\ d & e & f \end{vmatrix}$

$$= (-1)^{1+1} \cdot \vec{i} \cdot \begin{vmatrix} b & c \\ e & f \end{vmatrix}$$
$$+ (-1)^{1+2} \cdot \vec{j} \cdot \begin{vmatrix} a & c \\ d & f \end{vmatrix}$$
$$+ (-1)^{1+3} \cdot \vec{k} \cdot \begin{vmatrix} a & b \\ d & e \end{vmatrix}$$
$$= \vec{i} \cdot (bf - ce) - \vec{j}(af - cd) + \vec{k}(ae - bd)$$

① $\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \cdot \|\vec{v}\| \cdot \sin\theta$

is the area of ▱



② $\vec{u} \times \vec{v}$ is $90°$ to both $\vec{u}$ and $\vec{v}$

Example: $\vec{u} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, $\vec{v} = \begin{pmatrix} d \\ e \\ f \end{pmatrix}$, $\vec{w} = \begin{pmatrix} i \\ g \\ h \end{pmatrix}$

The tripple product is $\vec{u} \times \vec{v} \cdot \vec{w}$

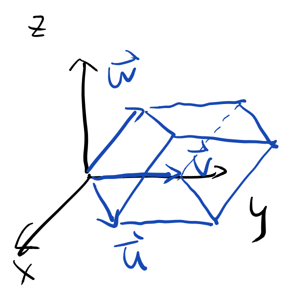
same

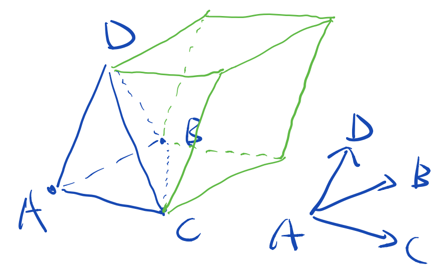
① First compute $\vec{u} \times \vec{v}$

Then compute $(\vec{u} \times \vec{v}) \cdot \vec{w}$

② $\vec{u} \times \vec{v} \cdot \vec{w} = \begin{vmatrix} a & b & c \\ d & e & f \\ i & g & h \end{vmatrix} = \begin{vmatrix} a & d & i \\ b & e & g \\ c & f & h \end{vmatrix}$

③ $|\vec{u} \times \vec{v} \cdot \vec{w}|$ is volume of parallelipiped





- $A \in \mathbb{R}^{4\times 4}$, $\det(A)$ is defined by cofactor expansion.

$$\det\begin{pmatrix} 1 & -1 & 2 & 3 \\ 3 & 4 & 1 & 2 \\ 2 & -5 & -3 & 8 \\ -6 & 2 & -4 & 1 \end{pmatrix} \overset{\text{(2nd row)}}{=} (-1)^{2+1} \cdot \boxed{3} \cdot \begin{vmatrix} -1 & 2 & 3 \\ -5 & -3 & 8 \\ 2 & -4 & 1 \end{vmatrix}$$

$$+ (-1)^{2+2} \cdot \boxed{4} \cdot \begin{vmatrix} 1 & 2 & 3 \\ 2 & -3 & 8 \\ -6 & -4 & 1 \end{vmatrix}$$

$$+(-1)^{2+3}\cdot\boxed{1}\cdot\begin{vmatrix}1 & -1 & 3\\ 2 & -5 & 8\\ -6 & 2 & 1\end{vmatrix}$$

$$+(-1)^{2+4}\cdot\boxed{2}\cdot\begin{vmatrix}1 & -1 & 2\\ 2 & -5 & -3\\ -6 & 2 & -4\end{vmatrix}$$

- Row/Col Ops

① Type I will change sign

$$\begin{vmatrix}a & b & c\\ d & e & f\\ i & g & h\end{vmatrix} = -\begin{vmatrix}a & c & b\\ d & f & e\\ i & h & g\end{vmatrix}$$

② Type II : factor a number out of any row/col

$$\begin{vmatrix}2 & 4 & 6\\ 1 & 2 & 3\\ 4 & 5 & 6\end{vmatrix} = 2\begin{vmatrix}1 & 2 & 3\\ 1 & 2 & 3\\ 4 & 5 & 6\end{vmatrix}$$

$$\begin{bmatrix}2 & 4 & 6\\ 2 & 2 & 2\\ 4 & 4 & 4\end{bmatrix} = 2\begin{bmatrix}1 & 2 & 3\\ 1 & 1 & 1\\ 2 & 2 & 2\end{bmatrix}$$

→ matrix

$$\begin{vmatrix}2 & 4 & 6\\ 2 & 2 & 2\\ 4 & 4 & 4\end{vmatrix} = 8\begin{vmatrix}1 & 2 & 3\\ 1 & 1 & 1\\ 2 & 2 & 2\end{vmatrix}$$

→ det

③ Type III : does not change det

Example:

$$(C1\cdot(-\tfrac{1}{2})+C4 \to C4)$$

$$\begin{vmatrix}2 & 0 & 0 & 1\\ 0 & 1 & 3 & -3\\ -2 & -3 & -5 & 2\\ 4 & -4 & 4 & -6\end{vmatrix} = \begin{vmatrix}2 & 0 & 0 & 0\\ 0 & 1 & 3 & -3\\ -2 & -3 & -5 & 3\\ 4 & -4 & 4 & -8\end{vmatrix}$$

$$= 4 \begin{vmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 3 & -3 \\ -2 & -3 & -5 & 3 \\ 1 & -1 & 1 & -2 \end{vmatrix} = 8 \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & -3 \\ -2 & -3 & -5 & 3 \\ 1 & -1 & 1 & -2 \end{vmatrix}$$

(cofactor expansion along first row)

$$= 8 \cdot (-1)^{1+1} \cdot 1 \cdot \begin{vmatrix} 1 & 3 & -3 \\ -3 & -5 & 3 \\ -1 & 1 & -2 \end{vmatrix}$$

$$= 8 \begin{vmatrix} 1 & 3 & -3 \\ -3 & -5 & 3 \\ -1 & 1 & -2 \end{vmatrix}$$

$$= 8 \begin{vmatrix} 1 & 3 & -3 \\ 0 & 4 & -6 \\ 0 & 4 & -5 \end{vmatrix}$$

$$= 8 \cdot (-1)^{1+1} \cdot 1 \cdot \begin{vmatrix} 4 & -6 \\ 4 & -5 \end{vmatrix}$$

$$= 8 \cdot (-20 + 24) = 32.$$

- Quick facts about det

① $\det(I) = 1$

② zero row/col $\Rightarrow$ $\det = 0$

③ same row/col $\Rightarrow$ $\det = 0$

④ $\det(A^T) = \det(A)$

⑤ $\det(AB) = \det(A)\det(B)$

⑥ $\det(A^{-1}) = \dfrac{1}{\det(A)}$

⑦ A is invertible $\Leftrightarrow \det(A) \neq 0$

Proof: We can turn A into RREF(A) by row ops.

Row/Col ops do not change det to zero.

So $\det(A) = 0 \Leftrightarrow \det(\text{RREF}) = 0$.

$$\Updownarrow$$

RREF has zero row

$$\Updownarrow$$

$\text{rank}(A) < n$.

$$\Updownarrow$$

A is not invertible.

- Inverse Matrix and det.

Let $C_{ij} = (-1)^{i+j} \cdot |\widetilde{A_{ij}}|$

If $A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$ is invertible, then

$$A^{-1} = \frac{1}{|A|} \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} = \begin{bmatrix} \boxed{1} & \boxed{4} & \square \\ \square & \boxed{2} & \square \\ \square & \square & \boxed{3} \end{bmatrix}$$

$$1 = A_{11} \cdot C_{11} + A_{12} \cdot C_{12} + A_{13} \cdot C_{13}$$

$$= A_{11} \cdot (-1)^{1+1} \cdot |\widetilde{A_{11}}|$$

$$+ A_{12} \cdot (-1)^{1+2} \cdot |\widetilde{A_{12}}|$$

$$+ A_{13} \cdot (-1)^{1+3} |\widetilde{A_{13}}| = \det(A)$$

④ $= A_{11} \cdot C_{21} + A_{12} \cdot C_{22} + A_{13} \cdot C_{23}$

$$= \underline{A_{11}} \cdot (-1)^{2+1} \cdot |\tilde{A}_{21}| + \underline{A_{12}} \cdot (-1)^{2+2} \cdot |\tilde{A}_{22}| + \underline{A_{13}} \cdot (-1)^{2+3} \cdot |\tilde{A}_{23}|$$

$$\boxed{A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}}$$

$$= \det \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{11} & A_{12} & A_{13} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} = 0.$$

- Linear System Sol and det (Cramer's Rule)

$$A\vec{x} = \vec{b}$$

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

If $\det(A) \neq 0$, then

$$\vec{x} = A^{-1}\vec{b}$$

$$= \frac{1}{|A|} \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\Rightarrow \quad x_1 = \frac{1}{|A|}\left(C_{11} \cdot b_1 + C_{21} \cdot b_2 + C_{31} \cdot b_3 \right)$$

$$= \frac{1}{|A|}\Big[b_1 \cdot (-1)^{1+1} \cdot |\widetilde{A_{11}}|$$

$$\boxed{A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}}$$

$$+ b_2 \cdot (-1)^{2+1} \cdot |\widetilde{A_{21}}|$$

$$+ b_3 \cdot (-1)^{3+1} \cdot |\widetilde{A_{31}}| \Big]$$

$$= \frac{1}{|A|} \cdot \begin{vmatrix} b_1 & A_{12} & A_{13} \\ b_2 & A_{22} & A_{23} \\ b_3 & A_{32} & A_{33} \end{vmatrix}$$

$$x_2 = \frac{1}{|A|} \begin{vmatrix} A_{11} & b_1 & A_{13} \\ A_{21} & b_2 & A_{23} \\ A_{31} & b_3 & A_{33} \end{vmatrix}$$

$$x_3 = \frac{1}{|A|} \begin{vmatrix} A_{11} & A_{12} & b_1 \\ A_{21} & A_{22} & b_2 \\ A_{31} & A_{32} & b_3 \end{vmatrix}$$

- Change of coordinates in integration

① Polar Coordinates $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

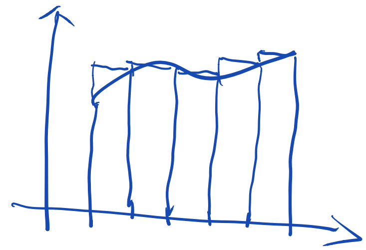
$$\iint_{x^2+y^2\le 1} f(x,y)\,dx\,dy = \int_0^{2\pi}\int_0^1 f(r,\theta)\,\boxed{r}\,dr\,d\theta$$

1) Jacobian Matrix is the first order derivative of a vector-valued multivariable function.

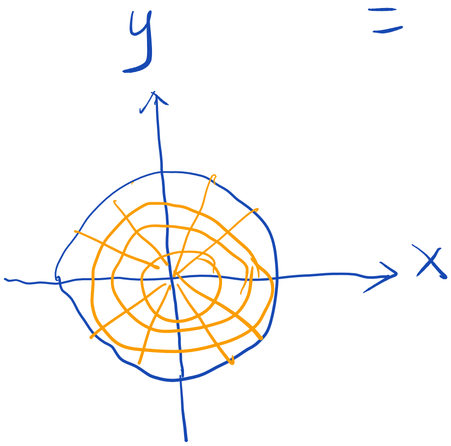
$$\vec{x}(r,\theta) = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$$

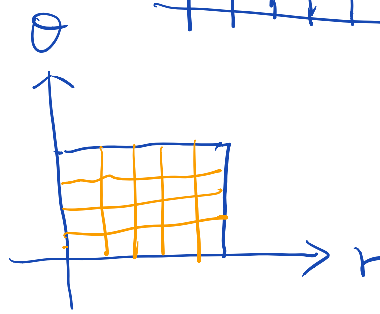
$$J = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

$$|J| = r\cos^2\theta + r\sin^2\theta = r$$



$$dx\,dy = |J|\,dr\,d\theta$$
$$= r\,dr\,d\theta$$





$$\approx |J|\Delta\theta\Delta r$$

$$\left(dx = \frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial \theta}d\theta \right)$$

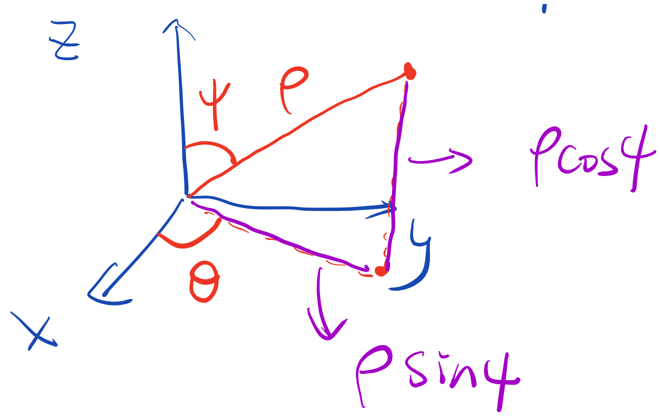
$$x = r\cos\theta \Rightarrow \Delta x \approx \frac{\partial x}{\partial r}\Delta r + \frac{\partial x}{\partial \theta}\Delta\theta$$

$$x = r\sin\theta \Rightarrow \Delta y \approx \frac{\partial y}{\partial r}\Delta r + \frac{\partial y}{\partial \theta}\Delta\theta$$

$$\left(dy = \frac{\partial y}{\partial r}dr + \frac{\partial y}{\partial \theta}d\theta \right)$$

② Spherical Coordinate

$$\begin{cases} x = \rho\sin\phi\cos\theta \\ y = \rho\sin\phi\sin\theta \\ z = \rho\cos\phi \end{cases}$$



$$dx\,dy\,dz = |J|\,d\rho\,d\psi\,d\theta$$

$$|J| = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \psi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \psi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \psi} & \frac{\partial z}{\partial \theta} \end{vmatrix}$$