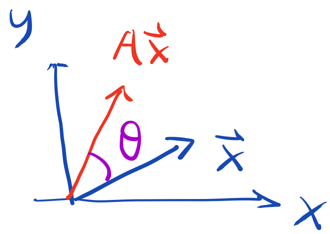
Chapter 6 Eigenvalues & Eigenvectors

- Any matrix $A \in \mathbb{R}^{n\times n}$ corresponds to a mapping L_A from $\mathbb{R}^n$ to $\mathbb{R}^n$:

$$L_A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$
$$\vec{x} \longmapsto A\vec{x}$$

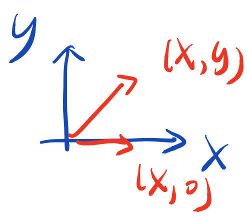
Example: ① $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$



$$L_A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta x - \sin\theta y \\ \sin\theta x + \cos\theta y \end{pmatrix}$$

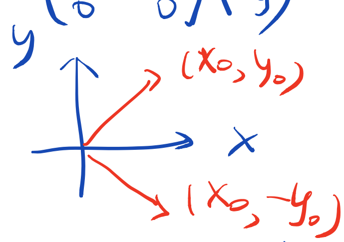
Rotation of angle θ counterclockwise

② $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$



$$L_A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix}$$

③ $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$



$$L_A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix}$$

- Question: For a square matrix $A \in \mathbb{R}^{n\times n}$,

  is there a nonzero vector $\vec{v}$ s.t.

  $A\vec{v}$ is parallel to $\vec{v}$?

  Which direction the mapping L_A does NOT change?

- Def (Eigenvalues & Eigenvectors)

  For $A \in \mathbb{R}^{n\times n}$, $\vec{v} \in \mathbb{R}^n$ or $\vec{v} \in \mathbb{C}^n$ is called an

  eigen-vector of A if $A\vec{v} = \lambda \vec{v}$ for

  some number λ and $\vec{v} \neq \vec{0}$.

  The number λ is called eigen-value of A,

  associated with the eigen-vector $\vec{v}$.

Example: ① If $\begin{cases} A\vec{v} = \vec{0} \\ \vec{v} \neq \vec{0} \end{cases}$, $\vec{v}$ is an eigenvector.

$$A\vec{v} = 0 \cdot \vec{v}$$

②
$$\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{A} \cdot \underbrace{\begin{pmatrix} x \\ 0 \end{pmatrix}}_{\vec{v}} = \begin{pmatrix} x \\ 0 \end{pmatrix} = \underbrace{1}_{\lambda} \cdot \underbrace{\begin{pmatrix} x \\ 0 \end{pmatrix}}_{\vec{v}}$$

- How to find Eigenvalues:

$$A\vec{v} = \lambda\vec{v} \iff A\vec{v} - \lambda\vec{v} = \vec{0}$$
$$\iff A\vec{v} - \lambda I \vec{v} = \vec{0}$$
$$\iff (A - \lambda I)\vec{v} = \vec{0}$$

$\iff (A-\lambda I)\vec{x}=\vec{0}$ has a nonzero sol.

$\iff A-\lambda I$ is singular (not invertible)

$\iff |A-\lambda I|=0$

$\det(A-\lambda I)$ is a polynomial of λ

Example: $A=\begin{pmatrix} 5 & 2 & 0 \\ 2 & 5 & 0 \\ -3 & 4 & 6 \end{pmatrix}$

$$\det(A-\lambda I) = \det\left[\begin{pmatrix} 5 & 2 & 0 \\ 2 & 5 & 0 \\ -3 & 4 & 6 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right]$$

$$= \begin{vmatrix} 5-\lambda & 2 & 0 \\ 2 & 5-\lambda & 0 \\ -3 & 4 & 6-\lambda \end{vmatrix}$$

$$= (6-\lambda)\cdot(-1)^{3+3}\cdot\begin{vmatrix} 5-\lambda & 2 \\ 2 & 5-\lambda \end{vmatrix}$$

$$= (6-\lambda)\cdot[(5-\lambda)^2-4]$$

$$= (6-\lambda)(\lambda^2-10\lambda+25-4)$$

$$= -(\lambda-6)(\lambda^2-10\lambda+21)$$

$$= -(\lambda-6)(\lambda-7)(\lambda-3) = 0$$

<u>Def</u> $\det(A-\lambda I)$ is a polynomial of degree n, called characteristic polynomial of $A \in \mathbb{R}^{n\times n}$. Its roots are eigenvalues of A.

- How to find eigenvectors:

  If λ is known, then solve $\vec{v}$ in $(A-\lambda I)\vec{v}=\vec{0}$.

Example: $A=\begin{pmatrix} 5 & 2 & 0 \\ 2 & 5 & 0 \\ -3 & 4 & 6 \end{pmatrix}$

$$|A-\lambda I| = -(\lambda-6)(\lambda-7)(\lambda-3) = 0$$

$$\Rightarrow \lambda_1=6, \quad \lambda_2=7, \quad \lambda_3=3$$

① Plug in $\lambda=6$ in $(A-\lambda I)\vec{v}=\vec{0}$

$$\begin{pmatrix} 5-\lambda & 2 & 0 \\ 2 & 5-\lambda & 0 \\ -3 & 4 & 6-\lambda \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 2 & 0 \\ 2 & -1 & 0 \\ -3 & 4 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} -1 & 2 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ -3 & 4 & 0 & 0 \end{array}\right)$$

$(A-\lambda I)\vec{v}=\vec{0}$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$\Rightarrow$ Sol is $\vec{v}=t\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \forall t\in\mathbb{R}$

So $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is an eigenvector

$t\begin{pmatrix}0\\0\\1\end{pmatrix}$ is an eigenvector if $t \neq 0$.

<u>Def</u> The subspace consisting of all eigenvectors for a particular eigenvalue λ and $\vec{0}$ is called eigenspace of λ.

Example: The eigenspace for $\lambda = 6$ is $\text{Span}\left\{\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$.

$\lambda_1 = 6$, $\lambda_2 = 7$, $\lambda_3 = 3$

② Plug in $\lambda = 7$ in $(A - \lambda I)\vec{v} = \vec{0}$

$$\begin{pmatrix} 5-\lambda & 2 & 0 \\ 2 & 5-\lambda & 0 \\ -3 & 4 & 6-\lambda \end{pmatrix}\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix} -2 & 2 & 0 \\ 2 & -2 & 0 \\ -3 & 4 & -1 \end{pmatrix}\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow \vec{v} = t\begin{pmatrix}1\\1\\1\end{pmatrix}$, $\forall t \in \mathbb{R}$

$\Rightarrow$ The eigenspace for $\lambda_2 = 7$ is $\text{Span}\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}\right\}$

Verification: $\begin{pmatrix} 5 & 2 & 0 \\ 2 & 5 & 0 \\ -3 & 4 & 6 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 7 \\ 7 \end{pmatrix} = 7\cdot\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

③ Plug in $\lambda = 3$ in $(A-\lambda I)\vec{v} = \vec{0}$

$$\begin{pmatrix} 5-\lambda & 2 & 0 \\ 2 & 5-\lambda & 0 \\ -3 & 4 & 6-\lambda \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ -3 & 4 & 3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -3/7 & 0 \\ 0 & 1 & 3/7 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \vec{v} = t\begin{pmatrix} 3/7 \\ -3/7 \\ 1 \end{pmatrix}, \quad \forall t \in \mathbb{R}$$

$t=7 \Rightarrow \begin{pmatrix} 3 \\ -3 \\ 7 \end{pmatrix}$ is one eigen-vector for $\lambda_3 = 3$

The eigen-space is $\mathrm{Span}\left\{\begin{bmatrix} 3 \\ -3 \\ 7 \end{bmatrix}\right\}$.

Verification $\begin{pmatrix} 5 & 2 & 0 \\ 2 & 5 & 0 \\ -3 & 4 & 6 \end{pmatrix}\begin{pmatrix} 3 \\ -3 \\ 7 \end{pmatrix} = \begin{pmatrix} 9 \\ -9 \\ 21 \end{pmatrix} = 3\cdot\begin{pmatrix} 3 \\ -3 \\ 7 \end{pmatrix}$