- $A \in \mathbb{R}^{n\times n}$, $\det(A) = |A|$ is a number
- Inverse Matrix and det. $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}, \quad A^{-1} = \frac{1}{|A|}\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

Let $C_{ij} = (-1)^{i+j} \cdot |\tilde{A}_{ij}|$

$\hookrightarrow$ cofactor matrix of A_{ij}

- Cramer's Rule for $A\vec{x} = b$ with $|A| \neq 0$:

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\begin{cases} x_1 = \dfrac{1}{|A|} \cdot \begin{vmatrix} b_1 & A_{12} & A_{13} \\ b_2 & A_{22} & A_{23} \\ b_3 & A_{32} & A_{33} \end{vmatrix} \\[2ex] x_2 = \dfrac{1}{|A|} \begin{vmatrix} A_{11} & b_1 & A_{13} \\ A_{21} & b_2 & A_{23} \\ A_{31} & b_3 & A_{33} \end{vmatrix} \\[2ex] x_3 = \dfrac{1}{|A|} \begin{vmatrix} A_{11} & A_{12} & b_1 \\ A_{21} & A_{22} & b_2 \\ A_{31} & A_{32} & b_3 \end{vmatrix} \end{cases}$$

- $\det(A-\lambda I)$ is a polynomial of degree n called characteristic polynomial.

Ex: $A = \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}$

$$|A-\lambda I| = \begin{vmatrix} 1-\lambda & 3 \\ 0 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda)$$
$$= \lambda^2 - 3\lambda + 2$$

- Roots of $\det(A-\lambda I)$ are eigenvalues of A.

- If a nonzero vector $\vec{v} \in \mathbb{R}^n$ satisfies $A\vec{v}$ is parallel to $\vec{v}$, $\vec{v}$ is eigenvector of A.

- $$A\vec{v} \parallel \vec{v} \iff A\vec{v} = \lambda\vec{v} \text{ for some number } \lambda$$
$$\iff A\vec{v} - \lambda\vec{v} = \vec{0}$$
$$\iff (A - \lambda I)\vec{v} = \vec{0}$$
$$\iff |A - \lambda I| = 0$$

- Eigenvector cannot be $\vec{0}$ but eigenvalues can be zero.

Ex: If $\vec{v}$ is a nonzero sol to $A\vec{x} = \vec{0}$,
then $A\vec{v} = 0 \cdot \vec{v} \Rightarrow \vec{v}$ is an eigenvector associated with eigenvalue $\lambda = 0$.

Example: Find eigenvalues of $A = \begin{pmatrix} -2 & -4 & 2 \\ -2 & 1 & 2 \\ 4 & 2 & 5 \end{pmatrix}$

Sol: $|A - \lambda I| = \begin{vmatrix} -2-\lambda & -4 & 2 \\ -2 & 1-\lambda & 2 \\ 4 & 2 & 5-\lambda \end{vmatrix}$

$$= (-1)^{1+1} \cdot (-2-\lambda) \cdot \begin{vmatrix} 1-\lambda & 2 \\ 2 & 5-\lambda \end{vmatrix}$$

$$+ (-1)^{2+1} \cdot (-2) \cdot \begin{vmatrix} -4 & 2 \\ 2 & 5-\lambda \end{vmatrix}$$

$$+ (-1)^{3+1} \cdot 4 \cdot \begin{vmatrix} -4 & 2 \\ 1-\lambda & 2 \end{vmatrix}$$

$$= -(\lambda+2)\left[(1-\lambda)(5-\lambda) - 4\right]$$

$$+ 2\left[-4(5-\lambda) - 4\right]$$

$$+ 4\left[-8 - 2(1-\lambda)\right]$$

$$= -(\lambda+2)\left[\lambda^2 - 6\lambda + 5 - 4\right]$$

$$+ 2\left[-20 + 4\lambda - 4\right]$$

$$+ 4\left[-8 - 2 + 2\lambda\right]$$

$$= -(\lambda+2)\left[\lambda^2 - 6\lambda + 1\right]$$

$$+ 2\left[4\lambda - 24\right]$$

$$+ 4\left[2\lambda - 10\right]$$

$$= -\left[\lambda^3 - 6\lambda^2 + \lambda + 2\lambda^2 - 12\lambda + 2\right]$$

$$+8\lambda - 48 + 8\lambda - 40$$

$$= -\lambda^3 + 4\lambda^2 + 27\lambda - 90$$

(By trial and error we find $\lambda = 3$ is a root)

try $\lambda = 0, \pm 1, \pm 2, \ldots$

$$= -(\lambda - 3)(\lambda^2 + a\lambda + b)$$

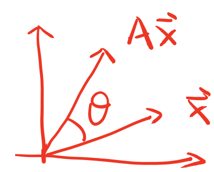
$$= -\lambda^3 + 3\lambda^2 - a\lambda^2 + 3a\lambda - b\lambda + 3b$$

$$\Rightarrow \begin{cases} 3 - a = 4 \Rightarrow a = -1 \\ 3a - b = 27 \\ 3b = -90 \Rightarrow b = -30 \end{cases}$$

$$= -(\lambda - 3)(\lambda^2 - \lambda - 30)$$

$$= -(\lambda - 3)(\lambda - 6)(\lambda + 5)$$

Remark: In practice, eigenvalues are computed by approximation algorithms on computers.



Example: $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

$$|A - \lambda I| = \begin{vmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{vmatrix}$$

$$= (\cos\theta - \lambda)^2 + \sin^2\theta$$

$$= \lambda^2 - 2\cos\theta\,\lambda + \cos^2\theta + \sin^2\theta$$

$$= \lambda^2 - 2\cos\theta\,\lambda + 1$$

Discriminant of a quadratic poly is

$$\Delta = 4\cos^2\theta - 4 = -4\sin^2\theta$$

So $\begin{cases} \Delta < 0 & \text{for } \theta \neq 0, \pi \Rightarrow \text{no real roots} \\ \Delta = 0 & \text{for } \begin{cases} \theta = 0 \Rightarrow \lambda = 1 \\ \theta = \pi \Rightarrow \lambda = -1 \end{cases} \end{cases}$

So if $\theta \neq 0$, then no real eigenvalues.

But we always have complex eigenvalues

$$\lambda = \frac{2\cos\theta \pm \sqrt{\Delta}}{2} = \frac{2\cos\theta \pm \sqrt{-4\sin^2\theta}}{2}$$

$$= \cos\theta \pm i\sin\theta \qquad \boxed{i = \sqrt{-1}}$$

This means A has complex eigenvectors.

We can also regard L_A as a mapping

$$\mathbb{R}^2 \to \mathbb{R}^2$$
$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto A\begin{pmatrix} x \\ y \end{pmatrix}$$

from $\mathbb{C}^2 \to \mathbb{C}^2$

$$L_A : \mathbb{C}^2 \to \mathbb{C}^2$$
$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto A\begin{pmatrix} x \\ y \end{pmatrix}$$

Theorem (Fundamental Theorem of Algebra)

A polynomial of degree n with real or complex coefficients always has n complex roots.

So for any $A \in \mathbb{R}^{n\times n}$, it always has n complex eigenvalues, but it may not have real eigenvalues. $(\lambda-1)^n$

Fun facts:

① [Abel Theorem] No root formula for polynomial of degree 5 and higher

② "roots" function in MATLAB can easily find accurate approximations to roots.

③ For any polynomial

$$p(t) = t^n + a_{n-1}t^{n-1} + a_{n-2}t^{n-2} + \cdots + a_1 t + a_0,$$

it has a companion matrix

$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{bmatrix}$$

s.t. $\quad |A - tI| = p(t)$

The "roots" function obtains approximation to roots by finding approximations to eigenvalues of its companion matrix via matrix algorithms

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Example: $A = \begin{pmatrix} 5 & 2 & 0 \\ 2 & 5 & 0 \\ -3 & 4 & 6 \end{pmatrix}$

$$\det(A - \lambda I) = -(\lambda - 6)(\lambda - 7)(\lambda - 3)$$

$\Rightarrow$ Three eigenvalues $\lambda_1 = 6$, $\lambda_2 = 7$, $\lambda_3 = 3$.

To find eigenvetors for each eigenvalue, plug in λ and solve the linear system

$$(A - \lambda I)\vec{v} = \vec{0}$$

Ex: plug in $\lambda = 6$ in $(A - \lambda I)\vec{v} = \vec{0}$

$$\begin{pmatrix} 5-\lambda & 2 & 0 \\ 2 & 5-\lambda & 0 \\ -3 & 4 & 6-\lambda \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 2 & 0 \\ 2 & -1 & 0 \\ -3 & 4 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow$ All solutions are $\vec{v} = t\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, $\forall t \in \mathbb{R}$

① So $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is an eigenvector.

② $t\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is an eigenvector if $t \neq 0$.

Eigenvector is not unique.

③ $\text{Span}\left\{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$ is called eigenspace for eigenvalue $\lambda = 6$.

- Consider $A \in \mathbb{R}^{6\times 6}$, assume

$$\det(A - \lambda I) = (\lambda - a)^2 (\lambda - b)(\lambda - c)^3$$

definition of alg mul $\begin{cases} \lambda = a \text{ has algebraic multiplicity } 2 \\ \lambda = b \text{ has algebraic multiplicity } 1 \\ \lambda = c \text{ has algebraic multiplicity } 3 \end{cases}$

The dimension of each eigen-space is called geometric multiplicity of each eigenvalue.

$\boxed{\text{Theorem}}$ $1 \leq$ Geometric Mul $\leq$ Algebraic Mul

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Ex: ① $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^3$$

$\Rightarrow$ A has only one eigenvalue $\lambda = 1$

with alg mul 3.

$$(A - \lambda I)\vec{v} = \vec{0}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow$ Eigen-space is $\text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

$\Rightarrow$ Geometrical Mul is 3.

② $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda)^2$$

$\Rightarrow \begin{cases} \lambda = 1 \text{ has alg mul } 1 \\ \lambda = 2 \text{ has alg mul } 2 \end{cases}$

Plug $\lambda=1$ into $(A-\lambda I)\vec{v}=\vec{0}$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow$ Eigenspace is $\text{Span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}$

$\Rightarrow$ $\lambda=1$ has Geo Mul 1

Plug $\lambda=2$ into $(A-\lambda I)\vec{v}=\vec{0}$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow$ Eigenspace is $\text{Span}\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}$

$\Rightarrow$ $\lambda=2$ has Geo Mul 1

Diagonalization of Matrices

For $A\in\mathbb{R}^{n\times n}$, assume it has n linearly independent eigenvectors (not always true)

$\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$ with eigenvalues

$\lambda_1, \lambda_2, \cdots, \lambda_n$ (could be repeated ones),

then $A\vec{v}_i=\lambda_i\vec{v}_i$ can imply diagonalization as follows:

$$A = \begin{pmatrix} 5 & 2 & 0 \\ 2 & 5 & 0 \\ -3 & 4 & 6 \end{pmatrix}, \quad \lambda_1 = 6, \ \lambda_2 = 7, \ \lambda_3 = 3$$

$$\vec{v}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \ \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \ \vec{v}_3 = \begin{pmatrix} 3 \\ -3 \\ 7 \end{pmatrix}$$

$$A\vec{v}_i = \lambda_i \vec{v}_i, \quad i = 1, 2, 3$$

$$A[\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3] = [\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3]\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$$\underbrace{\begin{pmatrix} 5 & 2 & 0 \\ 2 & 5 & 0 \\ -3 & 4 & 6 \end{pmatrix}}_{A}\underbrace{\begin{pmatrix} 0 & 1 & 3 \\ 0 & 1 & -3 \\ 1 & 1 & 7 \end{pmatrix}}_{V} = \underbrace{\begin{pmatrix} 0 & 1 & 3 \\ 0 & 1 & -3 \\ 1 & 1 & 7 \end{pmatrix}}_{V} \cdot \underbrace{\begin{pmatrix} 6 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 3 \end{pmatrix}}_{D}$$

$AV = VD \Rightarrow \underline{A = VDV^{-1}}$ diagonalization

Independent eigenvectors $\Rightarrow$ V has independent cols

$\Rightarrow$ col(V) is n-dimensional

$\Rightarrow$ rank$(V) = n$

$\Rightarrow$ V^{-1} exists

Find V^{-1} by Gaussian Elimination or Cofactor Matrix:

$$V^{-1} = \begin{pmatrix} -2/3 & -1/3 & 1 \\ 1/2 & 1/2 & 0 \\ 1/6 & -1/6 & 0 \end{pmatrix}$$

$A = VDV^{-1}$ is called diagonalization of A.

There are many applications of diagonalization.

Example: $A^2 = A \cdot A = VDV^{-1}VDV^{-1}$

$$= VD \cdot DV^{-1}$$

$$= VD^2V^{-1}$$

$$= V\begin{pmatrix} \lambda_1^2 & & \\ & \lambda_2^2 & \\ & & \lambda_3^2 \end{pmatrix}V^{-1}$$

$$A^n = V\begin{pmatrix} \lambda_1^n & & \\ & \lambda_2^n & \\ & & \lambda_3^n \end{pmatrix}V^{-1}$$

We can also define e^A by

$$e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots$$

$$= VIV^{-1} + VDV^{-1} + \frac{1}{2!}VD^2V^{-1} + \frac{1}{3!}VD^3V^{-1} + \cdots$$

$$= V\left[I + D + \frac{1}{2!}D^2 + \frac{1}{3!}D^3 + \cdots\right]V^{-1}$$

$$= V \begin{pmatrix} 1+\lambda_1+\frac{1}{2!}\lambda_1^2+\frac{1}{3!}\lambda_1^3+\cdots & 0 & 0 \\ 0 & 1+\lambda_2+\frac{1}{2!}\lambda_2^2+\frac{1}{3!}\lambda_2^3+\cdots & 0 \\ 0 & 0 & 1+\lambda_3+\frac{1}{2!}\lambda_3^2+\frac{1}{3!}\lambda_3^3+\cdots \end{pmatrix} V^{-1}$$

$$= V \begin{pmatrix} e^{\lambda_1} & & \\ & e^{\lambda_2} & \\ & & e^{\lambda_3} \end{pmatrix} V^{-1}$$

$$\underset{A}{\begin{pmatrix} 5 & 2 & 0 \\ 2 & 5 & 0 \\ -3 & 4 & 6 \end{pmatrix}} \underset{V}{\begin{pmatrix} 0 & 1 & 3 \\ 0 & 1 & -3 \\ 1 & 1 & 7 \end{pmatrix}} = \underset{V}{\begin{pmatrix} 0 & 1 & 3 \\ 0 & 1 & -3 \\ 1 & 1 & 7 \end{pmatrix}} \underset{D}{\begin{pmatrix} 6 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 3 \end{pmatrix}}$$

$$\Rightarrow \quad e^A = \begin{pmatrix} 0 & 1 & 3 \\ 0 & 1 & -3 \\ 1 & 1 & 7 \end{pmatrix} \begin{pmatrix} e^6 & 0 & 0 \\ 0 & e^7 & 0 \\ 0 & 0 & e^3 \end{pmatrix} \begin{pmatrix} 0 & 1 & 3 \\ 0 & 1 & -3 \\ 1 & 1 & 7 \end{pmatrix}^{-1}.$$

$A = VDV^{-1} \Rightarrow$

Diagonalization & Matrix Exponential can be used for solving linear differential equations such as

2nd order equation
$$\begin{cases} 2y''(t) + 3y'(t) + y(t) = 0 \\ y(0) = 1, \quad y'(0) = 2 \end{cases}$$

① 1st order equation $\begin{cases} y'(t) - a\,y(t) = 0 \\ y(0) = b \end{cases}$

$$y'(t) = a\,y(t)$$

$$\frac{y'(t)}{y(t)} = a$$

$$\int \frac{y'(t)}{y(t)}\,dt = \int a\,dt$$

$$\ln[y(t)] = at + C$$

$$y(t) = e^{at+C}$$

$$y(0) = b \Rightarrow e^{C} = b \Rightarrow y(t) = e^{at}\,b$$

$$y(t) = e^{at}\,y(0)$$

②
$$2y''(t) + 3y'(t) + y(t) = 0$$

$$\vec{u}(t) = \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix}, \quad \frac{d}{dt}\vec{u}(t) = \begin{pmatrix} y'(t) \\ y''(t) \end{pmatrix}$$

$$\Rightarrow \vec{u}'(t) = \begin{pmatrix} y'(t) \\ -\frac{3}{2}y'(t) - \frac{1}{2}y(t) \end{pmatrix}$$

$$\Rightarrow \frac{d}{dt}\begin{pmatrix} y(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{1}{2} & -\frac{3}{2} \end{pmatrix}\begin{pmatrix} y(t) \\ y'(t) \end{pmatrix}$$ first order system

$$\frac{d}{dt}\vec{u}(t) = A\,\vec{u}(t)$$

$$\vec{u}(t) = e^{At}\,\vec{u}(0)$$

$$\left(\frac{d}{dt}y(t) = a\,y(t) \Rightarrow y(t) = e^{at}\,y(0) \right)$$