

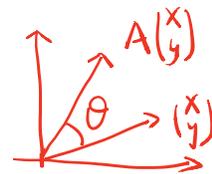
Today's Plan

- ① Review
- ② Diagonalization
- ③ Differential Equations

Review :

- $|A - \lambda I| = \det(A - \lambda I)$ is characteristic polynomial of A .
The roots of this polynomial are **eigenvalues of A** .
- For $A \in \mathbb{R}^{n \times n}$, Fundamental Theorem of Algebra implies $\det(A - \lambda I)$ has n complex roots (including repeated ones).
It may not have real roots.

Example: $A = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$



$$|A - \lambda I| = \lambda^2 - 2\cos\theta\lambda + 1 \quad \sim$$

Discriminant of a quadratic poly is

$$\Delta = 4\cos^2\theta - 4 = -4\sin^2\theta$$

$$\text{So } \begin{cases} \Delta < 0 & \text{for } \theta \neq 0, \pi \Rightarrow \text{no real roots} \\ \Delta = 0 & \text{for } \begin{cases} \theta = 0 \Rightarrow \lambda = 1 \\ \theta = \pi \Rightarrow \lambda = -1 \end{cases} \end{cases}$$

So no real eigenvalues (thus no real eigenvectors) for A if $\theta \neq 0, \pi$.

But A always has complex eigenvalues for any θ .

- If an eigenvalue λ is a root repeated m times for

the characteristic polynomial $p(t) = \det(A - tI)$,
then m is called **algebraic multiplicity of λ**

Example: $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$, $\det(A - \lambda I) = (2 - \lambda)^3$

1) The polynomial has 3 repeated roots

2) There is only one eigenvalue $\lambda = 2$ with algebraic multiplicity 3.

- Dimension of each eigen-space of an eigenvalue λ is called **geometric multiplicity of λ**

- Theorem: for each eigenvalue, $1 \leq \text{Geo Mul} \leq \text{Alg Mul}$.

Diagonalization

- Diagonalization of $A \in \mathbb{R}^{n \times n}$ means that we can find an invertible matrix $V \in \mathbb{R}^{n \times n}$ or $\mathbb{C}^{n \times n}$ and a diagonal matrix $D \in \mathbb{R}^{n \times n}$ or $\mathbb{C}^{n \times n}$ s.t.

$$A = V D V^{-1}.$$

- Not all matrices can be diagonalized.

- Assume $A = V D V^{-1}$ is true, want to see what it is equivalent to.

Let cols of V be $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$

Let diagonal entries of D be d_1, \dots, d_n

Take $n=3$ as an example:

$$V = [\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3]$$

$$D = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$$

$$A = V D V^{-1}$$

$$\Leftrightarrow AV = VD$$

$$\Leftrightarrow \begin{array}{c} \boxed{} \\ A \end{array} \begin{array}{|c|c|c|} \hline \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ \hline \end{array} = \begin{array}{|c|c|c|} \hline \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ \hline \end{array} \begin{array}{|c|c|c|} \hline d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \\ \hline \end{array} \begin{array}{c} \\ \\ \\ D \end{array}$$

$$\Leftrightarrow \begin{cases} A\vec{v}_1 = d_1\vec{v}_1 \\ A\vec{v}_2 = d_2\vec{v}_2 \\ A\vec{v}_3 = d_3\vec{v}_3 \end{cases} \begin{array}{l} A[\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3] \\ = [d_1\vec{v}_1 \ d_2\vec{v}_2 \ d_3\vec{v}_3] \quad \textcircled{1} \\ = [\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3] \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \quad \textcircled{2} \end{array}$$

Since $\vec{v}_i \neq \vec{0}$ (why?), \vec{v}_i is an eigenvector.

Theorem Let cols of V be $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$
 Let diagonal entries of D be d_1, \dots, d_n
 Then $A = VDV^{-1}$ if and only if
 $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ are linearly independent
 eigenvectors of A , with eigenvalues d_1, \dots, d_n .

Theorem If $\vec{v}_1, \dots, \vec{v}_m$ are eigenvectors of A
 with **distinct** eigenvalues $\lambda_1, \dots, \lambda_m$, then they
 are linearly independent

Remark: If we have n distinct eigenvalues
 for $A \in \mathbb{R}^{n \times n}$, then we only need to
 find one eigenvector \vec{v}_i for λ_i .

$$V = [\vec{v}_1 \ \vec{v}_2 \ \dots \ \vec{v}_n], \quad D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

$$\Rightarrow A = VDV^{-1}$$

Example:

$$A = \begin{pmatrix} 5 & 2 & 0 \\ 2 & 5 & 0 \\ -3 & 4 & 6 \end{pmatrix}, \quad \lambda_1 = 6, \quad \lambda_2 = 7, \quad \lambda_3 = 3$$

$$\vec{v}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} 3 \\ -3 \\ 1 \end{pmatrix}$$

$$V = \begin{pmatrix} 0 & 1 & 3 \\ 0 & 1 & -3 \\ 1 & 1 & 7 \end{pmatrix}, \quad D = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$\Rightarrow A = V D V^{-1}$$

$$= \begin{pmatrix} 0 & 1 & 3 \\ 0 & 1 & -3 \\ 1 & 1 & 7 \end{pmatrix} \begin{pmatrix} 6 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 0 & 1 & 3 \\ 0 & 1 & -3 \\ 1 & 1 & 7 \end{pmatrix}^{-1}$$

- If we don't have n distinct eigenvalues, then we simply collect basis vectors for all eigenspaces:

① If we can have n vectors, then diagonalizable

② If not, then A is not diagonalizable

Remark: If we allow complex eigenvalues,

then we have n eigenvalues including

repeated ones. This means

$$A \in \mathbb{R}^{6 \times 6}$$
$$(\lambda - a)^3 (\lambda - b)^2 (\lambda - c)$$

$$\sum_i (\text{Alg Mul of } \lambda_i) = n$$

$$\Rightarrow \sum_i (\text{Geo Mul of } \lambda_i) \leq n$$

↓
Number of basis vectors for all
eigenspaces.

So diagonalization \Leftrightarrow Geo Mul = Alg Mul
for all eigenvalues.

Example: $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda)^2$$

$$\Rightarrow \begin{cases} \lambda = 1 \text{ has alg mul } 1 \\ \lambda = 2 \text{ has alg mul } 2 \end{cases}$$

Eigenspace of $\lambda = 1$ is $\text{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

Eigenspace of $\lambda = 2$ is $\text{Span} \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

So A is not diagonalizable.

Example: $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1$$

No real roots \Rightarrow no real eigenvector

\Rightarrow real diagonalization is not possible

$\lambda = \pm i \Rightarrow$ two distinct complex eigenvalues

\Rightarrow complex diagonalization exists.

① Plug $\lambda = i$ into $(A - \lambda I)\vec{v} = \vec{0}$

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left[\begin{array}{cc|c} -i & -1 & 0 \\ 1 & -i & 0 \end{array} \right] \quad \begin{array}{l} v_2 = t \\ v_1 - i v_2 = 0 \\ \Rightarrow v_1 - it = 0 \Rightarrow v_1 = it \end{array}$$

$$\left[\begin{array}{cc|c} \boxed{1} & \boxed{-i} & 0 \\ 0 & 0 & 0 \end{array} \right] \Rightarrow \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} it \\ t \end{pmatrix} = t \begin{pmatrix} i \\ 1 \end{pmatrix}$$

\Rightarrow Eigenspace is $\text{Span} \left\{ \begin{bmatrix} i \\ 1 \end{bmatrix} \right\} = \left\{ t \begin{bmatrix} i \\ 1 \end{bmatrix}, \forall t \in \mathbb{R} \right\}$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ i \end{bmatrix} = i \begin{bmatrix} i \\ 1 \end{bmatrix}$$

② $\lambda = -i$ has eigenspace $\text{Span} \left\{ \begin{bmatrix} -i \\ 1 \end{bmatrix} \right\}$

$$\text{So } A = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}^{-1}$$

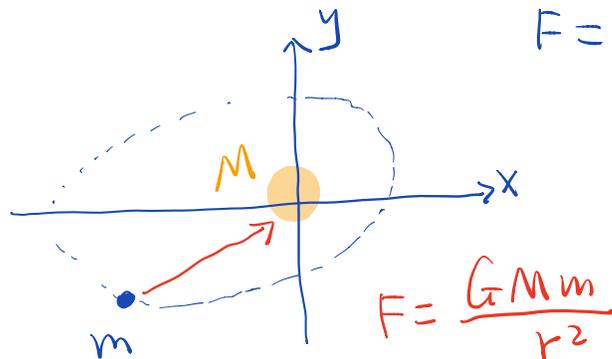
• Ordinary Differential Equations (ODE)

Equations like $y''(t) = -y'(t) + y(t)$ is called ODE

The variable t usually means time in many applications.

Example: Planet Motion by Newton's Law

$$F = ma$$



In a 2D Universe, the position of planet

is $\vec{x}(t) = (x(t), y(t))$

$\vec{v}(t) = \vec{x}'(t) = (x'(t), y'(t))$ is the velocity

$\vec{a}(t) = \vec{v}'(t) = \vec{x}''(t) = (x''(t), y''(t))$ is the acceleration.

$$F = \frac{GMm}{r^2} = \frac{GMm}{\|\vec{x}\|^2} = \frac{GMm}{[x(t)]^2 + [y(t)]^2}$$

Directional force:

$$\vec{F} = F \cdot \left(-\frac{\vec{x}}{\|\vec{x}\|}\right) = \frac{-GMm}{\|\vec{x}\|^3} \vec{x}$$

$$\vec{F} = m\vec{a} \Rightarrow -\frac{GMm}{\|\vec{x}\|^3} \vec{x} = m(x''(t), y''(t))$$

$$\Rightarrow \begin{cases} x''(t) = \frac{-GM}{[x(t)^2 + y(t)^2]^{3/2}} x(t) \\ y''(t) = \frac{-GM}{[x(t)^2 + y(t)^2]^{3/2}} y(t) \end{cases}$$

This is a nonlinear ODE,
for which we need to understand linear ODE first.

- Given a second order ODE, we can convert it to a first order system:

$$y''(t) + a y'(t) + b y(t) = 0$$

$$\frac{d}{dt} \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} y'(t) \\ y''(t) \end{pmatrix} = \begin{pmatrix} y' \\ -ay' - by \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix} \begin{pmatrix} y \\ y' \end{pmatrix}$$

- Given a third order ODE, we can convert it to a first order system:

$$y'''(t) + a y''(t) + b y'(t) + c y(t) = 0$$

$$\frac{d}{dt} \begin{pmatrix} y \\ y' \\ y'' \end{pmatrix} = \begin{pmatrix} y' \\ y'' \\ -ay'' - by' - cy \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -c & -b & -a \end{pmatrix} \begin{pmatrix} y \\ y' \\ y'' \end{pmatrix}$$

- So from now on, we focus on a first order ODE system

$$\frac{d}{dt} \vec{u}(t) = A \vec{u}(t), \quad \vec{u}(t) \in \mathbb{R}^n, \quad A \in \mathbb{R}^{n \times n}$$

$$\frac{d}{dt} \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}$$

- The simplest case is $n=1$:

$$\frac{d}{dt} u(t) = a u(t)$$

$$\frac{u'(t)}{u(t)} = a$$

$$\int \frac{u'(t)}{u(t)} dt = \int a dt$$

$$\ln u(t) = at + C$$

$$\Rightarrow u(t) = e^{at} \cdot e^c$$

$$\text{Plug in } t=0 \Rightarrow e^c = u(0)$$

$$\Rightarrow u(t) = e^{at} u(0)$$

$$\bullet \quad \frac{d}{dt} \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - \lambda - 2 = (\lambda+1)(\lambda-2)$$

$$\lambda = -1 \text{ has eigenspace } \text{Span} \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$$

$$\lambda = 2 \text{ has eigenspace } \text{Span} \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$$

$$V = \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix}, \quad D = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$$

$$V^{-1} = \frac{1}{-3} \begin{bmatrix} 1 & -2 \\ -1 & -1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix}$$

$$A = V D V^{-1} = \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix}^{-1}$$

$$\frac{d}{dt} \vec{u}(t) = A \vec{u}(t)$$

$$\frac{d}{dt} \vec{u}(t) = V D V^{-1} \vec{u}(t)$$

$$V^{-1} \frac{d}{dt} \vec{u}(t) = D V^{-1} \vec{u}(t)$$

$$\frac{d}{dt} [V^{-1} \vec{u}(t)] = D \cdot [V^{-1} \vec{u}(t)]$$

Change of variable $\vec{w}(t) = V^{-1} \vec{u}(t)$

$$\begin{aligned} \begin{pmatrix} w_1(t) \\ w_2(t) \end{pmatrix} &= \frac{1}{3} \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix} \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} \\ &= \begin{bmatrix} -\frac{1}{3} u_1(t) + \frac{2}{3} u_2(t) \\ \frac{1}{3} u_1(t) + \frac{1}{3} u_2(t) \end{bmatrix} \end{aligned}$$

$$\frac{d}{dt} \vec{w}(t) = D \vec{w}(t)$$

$$\begin{pmatrix} \frac{d}{dt} w_1(t) \\ \frac{d}{dt} w_2(t) \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} w_1(t) \\ w_2(t) \end{pmatrix}$$

$$\Rightarrow \begin{cases} \frac{d}{dt} w_1(t) = (-1) \cdot w_1(t) \\ \frac{d}{dt} w_2(t) = 2 \cdot w_2(t) \end{cases}$$

$$\Rightarrow \begin{cases} w_1(t) = e^{-t} w_1(0) \\ w_2(t) = e^{2t} w_2(0) \end{cases}$$

$$\Rightarrow \vec{w}(t) = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{pmatrix} \vec{w}(0)$$

$$\vec{w}(t) = V^{-1} \vec{u}(t)$$

$$\Rightarrow V^{-1} \vec{u}(t) = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{pmatrix} V^{-1} \vec{u}(0)$$

$$\Rightarrow \vec{u}(t) = V \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{pmatrix} V^{-1} \vec{u}(0)$$

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} = V \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} V^{-1}$$

$A t = tA = \begin{pmatrix} t & 2t \\ t & 0 \end{pmatrix}$ has eigenvalues $-t, 2t$
with same eigenvectors V .

$$\text{So } e^{At} = V \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{pmatrix} V^{-1}$$

$$\Rightarrow \vec{u}(t) = e^{At} \vec{u}(0)$$

How matrix exponential is defined?

$$\text{If } A = V D V^{-1} = V \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} V^{-1}$$

$$\text{then } A^2 = A \cdot A = V D V^{-1} V D V^{-1}$$

$$= V D \cdot D V^{-1}$$

$$= V D^2 V^{-1}$$

$$= V \begin{pmatrix} \lambda_1^2 & & \\ & \lambda_2^2 & \\ & & \lambda_3^2 \end{pmatrix} V^{-1}$$

$$A^n = V \begin{pmatrix} \lambda_1^n & & \\ & \lambda_2^n & \\ & & \lambda_3^n \end{pmatrix} V^{-1}$$

We obtain e^A by

$$e^A = I + A + \frac{1}{2!} A^2 + \frac{1}{3!} A^3 + \dots$$

$$= VIV^{-1} + VDV^{-1} + \frac{1}{2!} VD^2V^{-1} + \frac{1}{3!} VD^3V^{-1} + \dots$$

$$= V \left[I + D + \frac{1}{2!} D^2 + \frac{1}{3!} D^3 + \dots \right] V^{-1}$$

$$V \begin{pmatrix} 1 + \lambda_1 + \frac{1}{2!} \lambda_1^2 + \frac{1}{3!} \lambda_1^3 + \dots & 0 & 0 \\ 0 & 1 + \lambda_2 + \frac{1}{2!} \lambda_2^2 + \frac{1}{3!} \lambda_2^3 + \dots & 0 \\ 0 & 0 & 1 + \lambda_3 + \frac{1}{2!} \lambda_3^2 + \frac{1}{3!} \lambda_3^3 + \dots \end{pmatrix} V^{-1}$$

$$= V \begin{pmatrix} e^{\lambda_1} & & \\ & e^{\lambda_2} & \\ & & e^{\lambda_3} \end{pmatrix} V^{-1}$$

$$A = V D V^{-1} = V \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} V^{-1}$$

$$tA = t V D V^{-1} = V \cdot tD \cdot V^{-1} = V \begin{pmatrix} -t & 0 \\ 0 & 2t \end{pmatrix} V^{-1}$$