Systems of differential equations

$$\frac{d}{dt}\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}$$

They arise in many applications

① Dynamical systems for planet/satellite motion

$$\begin{cases} x''(t) = \dfrac{-GM}{[x(t)^2+y(t)^2]^{3/2}}\, x(t) \\[2ex] y''(t) = \dfrac{-GM}{[x(t)^2+y(t)^2]^{3/2}}\, y(t) \end{cases}$$

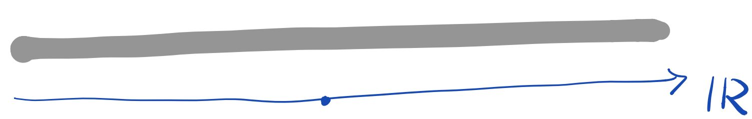
This is a nonlinear ODE,

for which we need to unstand linear ODE first.

② Partial Differential Equations

Example: The heat equation describes how heat diffuses on a given a region.

Consider a long thin metal bar



Let $u(x,t)$ be the temperature of the bar at location x and time t.

The heat equation is given as

$$\frac{\partial}{\partial t} u(x,t) = c \frac{\partial^2}{\partial x^2} u(x,t)$$

or $$u_t = c\, u_{xx}$$

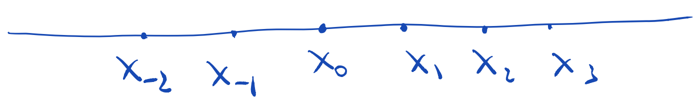
where c is a constant for thermal diffusivity of the metal bar.

An initial value problem is $\begin{cases} u_t = c\, u_{xx} \\ u(x,0) = f(x) \end{cases}$

for a given initial temperature $f(x)$.

To approximate the solution, we consider grid points x_i with uniform spacing Δx

$$x_i = i\Delta x$$



Let $u_i(t) = u(x_i, t)$, denoted by u_i.

The second order derivative can be approximated by

$$u_{xx}(x_i, t) \approx \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2}$$

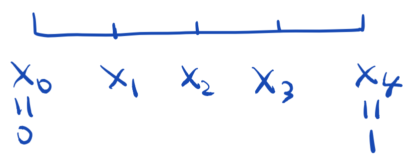
because

$$u(x_{i+1}) = u(x_i + \Delta x) \approx u(x_i) + \Delta x\, u_x(x_i) + \frac{\Delta x^2}{2} u_{xx}(x_i) + O(\Delta x^3)$$

$$u(x_{i-1}) = u(x_i - \Delta x) \approx u(x_i) - \Delta x\, u_x(x_i) + \frac{\Delta x^2}{2} u_{xx}(x_i) + O(\Delta x^3)$$

$$\Rightarrow \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2} \approx u_{xx}(x_i) + O(\Delta x)$$

Suppose we consider a finite interval $x \in [0,1]$ and impose boundary condition $u(0,t) = u(1,t) = 0$ which means the bar is cold on two ends.



$\Delta x = \frac{1}{4}$

$x_i = \Delta x \cdot i = \frac{i}{4}$

$u_0(t) = 0$ $\qquad$ $u_t = c\, u_{xx}$

$u_4(t) = 0$

$$\frac{d}{dt} u_1(t) = c\frac{u_0 - 2u_1 + u_2}{\Delta x^2} = c\,\frac{-2u_1 + u_2}{\Delta x^2}$$

$$\frac{d}{dt} u_2(t) = c\frac{u_1 - 2u_2 + u_3}{\Delta x^2}$$

$$\frac{d}{dt} u_3(t) = c\frac{u_2 - 2u_3 + u_4}{\Delta x^2} = c\,\frac{u_2 - 2u_3}{\Delta x^2}$$

$$\Rightarrow \frac{d}{dt}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \frac{c}{\Delta x^2}\begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

More points / Smaller Δx $\Rightarrow$
- Better approximation
- More equations
- Larger system/matrix

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① $y'''(t) - 2y''(t) - 3y'(t) + y(t) = 0$, $\vec{u} = \begin{pmatrix} y \\ y' \\ y'' \end{pmatrix}$

② $\frac{d}{dt}\vec{u}(t) = A\vec{u}(t)$, $\vec{u}(t) \in \mathbb{R}^3$, $A \in \mathbb{R}^{3\times 3}$ $\qquad$ (*)

Consider all solutions to (*), which form a subset of $V = \left\{ \vec{u} = \begin{pmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{pmatrix} : u_1(t), u_2(t), u_3(t) \text{ are differentiable} \right\}$

① V is an abstract vector space

1) $+$ and $\cdot$ are defined

2) closed under two ops

② Let $\vec{v}(t)$, $\vec{w}(t)$ be solutions to $(*)$

$$\frac{d}{dt}\vec{v}(t) = A\vec{v}(t)$$

$$\frac{d}{dt}\vec{w}(t) = A\vec{w}(t)$$

$$\Rightarrow \begin{cases} \frac{d}{dt}(\vec{v}+\vec{w}) = A(\vec{v}+\vec{w}) \\ \frac{d}{dt}[a\vec{v}(t)] = A[a\vec{v}(t)], \quad a \in \mathbb{R} \end{cases}$$

If S is the set of all solutions to $(*)$,

then S is closed under $+$ and $\cdot$

So $S \subseteq V$ is a subspace.

③ The dimension of S is less than or equal to 3.

If $\vec{v}_1(t), \vec{v}_2(t), \vec{v}_3(t), \vec{v}_4(t) \in S$,

$$a\vec{v}_1(t) + b\vec{v}_2(t) + c\vec{v}_3(t) + d\vec{v}_4(t) = \vec{0}$$

$$\Rightarrow a\vec{v}_1(t_0) + b\vec{v}_2(t_0) + c\vec{v}_3(t_0) + d\vec{v}_4(t_0) = \vec{0}$$

for any fixed t_0

for example, $t_0 = 1$

$$a\vec{v}_1(1) + b\vec{v}_2(1) + c\vec{v}_3(1) + d\vec{v}_4(1) = \vec{0}$$

$$\vec{v}_1(1), \vec{v}_2(t), \vec{v}_3(1), \vec{v}_4(1) \in \mathbb{R}^3$$

$\Rightarrow$ Nonzero $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$ exists $\Rightarrow$ Dependence

④ If $\vec{v}$ is an eigenvector of A with eigenvalue λ,

then $\vec{u}(t) = e^{\lambda t}\vec{v}$ is a sol to $(*)$

$$LHS = \frac{d}{dt}\vec{u}(t) = \lambda e^{\lambda t}\vec{v}$$

$$RHS = A\vec{u}(t) = A e^{\lambda t}\vec{v} = e^{\lambda t} A\vec{v} = e^{\lambda t}\lambda\vec{v}$$

⑤ **Theorem**

Then $A = VDV^{-1}$ if and only if

$\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are linearly independent

eigenvectors of A, with eigenvalues $d_1, \ldots, d_n$.

- A being diagonalizable means dimensions of all eigenspaces sum to n.

Next lecture, we will see that

real symmetric $(A = A^T)$ and real skew-symmetric $(A = -A^T)$

are always diagonalizable.

$$\begin{pmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

Assume $A \in \mathbb{R}^{3\times 3}$ is diagonalizable, then

we have $\begin{cases} A\vec{v}_1 = \lambda_1 \vec{v}_1 \\ A\vec{v}_2 = \lambda_2 \vec{v}_2 \\ A\vec{v}_3 = \lambda_3 \vec{v}_3 \end{cases}$

and $\begin{cases} \vec{v}_1, \vec{v}_2, \vec{v}_3 \text{ are independent.} \\ \lambda_1, \lambda_2, \lambda_3 \text{ may or may not be the same.} \end{cases}$

We get 3 sols to $(*)$:

$$\begin{cases} \vec{u}_1(t) = e^{\lambda_1 t}\vec{v}_1 \\ \vec{u}_2(t) = e^{\lambda_2 t}\vec{v}_2 \\ \vec{u}_3(t) = e^{\lambda_3 t}\vec{v}_3 \end{cases}$$

Check independence:

$$a\vec{u}_1(t) + b\vec{u}_2(t) + c\vec{u}_3(t) = \vec{0}$$

$$\Rightarrow a\vec{u}_2(0) + b\vec{u}_2(0) + c\vec{u}_3(0) = \vec{0}$$

$$\Rightarrow a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_3 = \vec{0}$$

$$\Rightarrow a = b = c = 0$$

$$\Rightarrow 3 \text{ independent sols to } (*)$$

Recall the solution set S is at most 3-dim

So $\{\vec{u}_1(t), \vec{u}_2(t), \vec{u}_3(t)\}$ is a basis of S.

For any $\vec{w}(t) \in S$, we have

$$\vec{w} = a\vec{u}_1 + b\vec{u}_2 + c\vec{u}_3$$

$$a\vec{u}_1 + b\vec{u}_2 + c\vec{u}_3 = \vec{w}$$

$$\begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \vec{w}$$

$$\begin{bmatrix} e^{\lambda_1 t}\vec{v}_1 & e^{\lambda_2 t}\vec{v}_2 & e^{\lambda_3 t}\vec{v}_3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} w_1(t) \\ w_2(t) \\ w_3(t) \end{bmatrix}$$

Plug in $t=0$

$$\Rightarrow \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} w_1(0) \\ w_2(0) \\ w_3(0) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix}^{-1} \begin{bmatrix} w_1(0) \\ w_2(0) \\ w_3(0) \end{bmatrix}$$

$$= V^{-1}\vec{w}(0)$$

$$\Rightarrow \vec{w} = a\vec{u}_1 + b\vec{u}_2 + c\vec{u}_3$$

$$= \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$= \begin{bmatrix} e^{\lambda_1 t}\vec{v}_1 & e^{\lambda_2 t}\vec{v}_2 & e^{\lambda_3 t}\vec{v}_3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$= \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 \\ 0 & 0 & e^{\lambda_3 t} \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$= \underbrace{[\vec{v}_1 \ \ \vec{v}_2 \ \ \vec{v}_3]}_{V} \begin{bmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 \\ 0 & 0 & e^{\lambda_3 t} \end{bmatrix} \underbrace{[\vec{v}_1 \ \ \vec{v}_2 \ \ \vec{v}_3]^{-1}}_{V^{-1}} \begin{bmatrix} w_1(0) \\ w_2(0) \\ w_3(0) \end{bmatrix}$$

$$= e^{At} \vec{w}(0)$$

Example: Initial Value Problem (IVP)

$$\begin{cases} y''(t) - y'(t) - 2y(t) = 0 \\ y(0) = 0 , \quad y'(0) = 1 \end{cases}$$

$$\frac{d}{dt}\begin{pmatrix} y \\ y' \end{pmatrix} = \begin{pmatrix} y' \\ y'' \end{pmatrix} = \begin{pmatrix} y' \\ y' + 2y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} y \\ y' \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ 2 & 1-\lambda \end{vmatrix} = -\lambda + \lambda^2 - 2 = (\lambda - 2)(\lambda + 1)$$

$$\Rightarrow \lambda = -1 , 2$$

$$\vec{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}$$

$$V = \begin{pmatrix} -1 & \frac{1}{2} \\ 1 & 1 \end{pmatrix}$$

$$V^{-1} = \frac{1}{-1 - \frac{1}{2}} \begin{pmatrix} 1 & -\frac{1}{2} \\ -1 & -1 \end{pmatrix} = \frac{2}{3} \begin{pmatrix} -1 & \frac{1}{2} \\ 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} y(t) \\ y'(t) \end{pmatrix} = V \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{pmatrix} V^{-1} \begin{pmatrix} y(0) \\ y'(0) \end{pmatrix}$$

$$\begin{pmatrix} y(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} -1 & \frac{1}{2} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{pmatrix} \frac{2}{3} \begin{pmatrix} -1 & \frac{1}{2} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & \frac{1}{2} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{pmatrix} \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}$$

$$= \begin{pmatrix} -1 & \frac{1}{2} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{3} e^{-t} \\ \frac{2}{3} e^{2t} \end{pmatrix}$$

$$\Rightarrow \begin{cases} y(t) = -\frac{1}{3} e^{-t} + \frac{1}{3} e^{2t} \\ y'(t) = \frac{1}{3} e^{-t} + \frac{2}{3} e^{2t} \end{cases}$$

$$\begin{cases} y''(t) - y'(t) - 2y(t) = 0 \\ y(0) = 0, \quad y'(0) = 1 \end{cases}$$

$$y'' = -\frac{1}{3} e^{-t} + \frac{4}{3} e^{2t}$$

$$-y' = -\frac{1}{3} e^{-t} - \frac{2}{3} e^{2t}$$

$$-2y = +\frac{2}{3} e^{-t} - \frac{2}{3} e^{2t}$$

$y_1(t) \qquad y_2(t) \qquad \begin{pmatrix} y_1 \\ y_1' \end{pmatrix} \quad \begin{pmatrix} y_2 \\ y_2' \end{pmatrix}$

$$\begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = y_1(t) y_2'(t) - y_2(t) y_1'(t) \neq 0$$