Remarks:

① For $A \in \mathbb{R}^{n\times n}$, eigenvectors are not unique for a given eigenvalue.

Example: If $A \in \mathbb{R}^{3\times 3}$ satisfies $A\begin{bmatrix}1\\1\\1\end{bmatrix} = 2\cdot\begin{bmatrix}1\\1\\1\end{bmatrix}$

$\forall a \neq 0$, $a\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}a\\a\\a\end{bmatrix}$ is an eigenvector of A for the eigenvalue $\lambda = 2$.

② $A \in \mathbb{R}^3$, $V = [\vec{v}_1\ \vec{v}_2\ \vec{v}_3]$, $D = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$

digonalization $A = VDV^{-1}$

$\Leftrightarrow$ $AV = VD$ for an invertible V

$\Leftrightarrow$ $A[\vec{v}_1\ \vec{v}_2\ \vec{v}_3] = [\vec{v}_1\ \vec{v}_2\ \vec{v}_3]\begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$

$$\Leftrightarrow \begin{cases} A\vec{v}_1 = d_1\vec{v}_1 \\ A\vec{v}_2 = d_2\vec{v}_2 \\ A\vec{v}_3 = d_3\vec{v}_3 \end{cases}$$

$\Leftrightarrow$ A has n linearly independent eigenvectors

$\Leftrightarrow$ $\begin{cases} \text{A has n eigenvalues including repeated ones} \\ \text{For all eigenspace: Geo Mul = Alg Mul} \end{cases}$

③ If A is diagonalizable, then there are many matrices V, D satisfying $A = VDV^{-1}$

$$A[\vec{v}_1\ \vec{v}_2\ \vec{v}_3] = [\vec{v}_1\ \vec{v}_2\ \vec{v}_3]\begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$$

$$A[\vec{v}_2\ \vec{v}_3\ \vec{v}_1] = [\vec{v}_2\ \vec{v}_3\ \vec{v}_1]\begin{bmatrix} d_2 & 0 & 0 \\ 0 & d_3 & 0 \\ 0 & 0 & d_1 \end{bmatrix}$$

$$A[\vec{v}_2\ 2\cdot\vec{v}_3\ \vec{v}_1] = [\vec{v}_2\ 2\cdot\vec{v}_3\ \vec{v}_1]\begin{bmatrix} d_2 & 0 & 0 \\ 0 & d_3 & 0 \\ 0 & 0 & d_1 \end{bmatrix}$$

④ Basis vectors for each eigen-space are not unique

⑤ Any nonzero vector in eigen-space is an eigenvector.

The following real matrices are always diagonalizable:

① Real Symmetric Matrix $A = A^T$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 6 & 7 \\ 3 & 6 & 8 & 9 \\ 4 & 7 & 9 & 10 \end{pmatrix}$$

② Real Skew-Symmetric $A^T = -A$

$$\begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{pmatrix}$$

③ Normal Matrix $AA^T = A^TA$

Theorem If $A \in \mathbb{R}^{n\times n}$ satisfies $A = A^T$, then

① A has n real eigenvalues including repeated ones

② A has n orthogonal real eigenvectors, i.e., all eigenspaces are orthogonal to one another.

Proof: ① $\det(A - \lambda I)$ has n complex roots including repeated ones.

Each distinct eigenvalue $\lambda = a + ib$ has at least one eigenvector $\vec{v}$

$$A\vec{v} = \lambda \vec{v}$$

$$\overline{a+ib} = a - ib$$

Take conjugate:

$$\overline{A\vec{v}} = \overline{\lambda \vec{v}}$$

$$\bar{A}\,\overline{\vec{v}} = \bar{\lambda}\,\overline{\vec{v}}$$

$(\bar{A} = A)$ $\quad A\overline{\vec{v}} = \bar{\lambda}\,\overline{\vec{v}}$

$$(AB)^T = B^T A^T$$

Take transpose:

$$\overline{\vec{v}}^T A^T = \overline{\vec{v}}^T \bar{\lambda}$$

$(A^T = A)$ $\quad \overline{\vec{v}}^T A = \overline{\vec{v}}^T \bar{\lambda}$

$$\overline{\vec{v}}^T A \vec{v} = \overline{\vec{v}}^T \bar{\lambda} \vec{v} \quad (*)$$

$$A\vec{v} = \lambda\vec{v} \Rightarrow \overline{\vec{v}}^T A\vec{v} = \overline{\vec{v}}^T \lambda\vec{v} \quad (**)$$

$$\left.\begin{array}{l}(*)\\(**)\end{array}\right\} \Rightarrow \overline{\vec{v}}^T \bar{\lambda}\vec{v} = \overline{\vec{v}}^T \lambda\vec{v}$$

$$\Rightarrow \bar{\lambda}\,\overline{\vec{v}}^T\vec{v} = \lambda\,\overline{\vec{v}}^T\vec{v}$$

$$\Rightarrow (\bar{\lambda}-\lambda)\underbrace{\overline{\vec{v}}^T\vec{v}}_{} = 0$$

$$[\bar{v}_1\ \bar{v}_2\ \bar{v}_3]\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix} = \bar{v}_1 v_1 + \bar{v}_2 v_2 + \bar{v}_3 v_3$$

$$= |v_1|^2 + |v_2|^2 + |v_3|^2 > 0$$

$$\Rightarrow \bar{\lambda} = \lambda \Rightarrow \lambda \in \mathbb{R}$$

② We already know n complex eigenvalues are real, so all eigenvectors are real.

Want to show $\left.\begin{array}{l}A\vec{v}_1 = \lambda_1\vec{v}_1\\ A\vec{v}_2 = \lambda_2\vec{v}_2\\ \lambda_1 \neq \lambda_2\end{array}\right\} \Rightarrow \vec{v}_1 \perp \vec{v}_2$

$$A\vec{v}_1 = \lambda_1\vec{v}_1$$

Take dot product with $\vec{v}_2$

$$\langle A\vec{v}_1, \vec{v}_2\rangle = \langle \lambda_1\vec{v}_1, \vec{v}_2\rangle$$

$$(\vec{v}_2^T A)\vec{v}_1 = \lambda_1\langle \vec{v}_1, \vec{v}_2\rangle$$

$(ABC)^T = C^T B^T A^T$ $\qquad (A^T\vec{v}_2)^T\vec{v}_1 = \lambda_1\langle\vec{v}_1,\vec{v}_2\rangle$

$A^T = A$ $\qquad (A\vec{v}_2)^T\vec{v}_1 = \lambda_1\langle\vec{v}_1,\vec{v}_2\rangle$

$$(\lambda_2 \vec{v}_2)^T \vec{v}_1 = \lambda_1 < \vec{v}_1 , \vec{v}_2 >$$

$$\lambda_2 \vec{v}_2^T \vec{v}_1 = \lambda_1 < \vec{v}_1 , \vec{v}_2 >$$

$$\lambda_2 < \vec{v}_1 , \vec{v}_2 > = \lambda_1 < \vec{v}_1 , \vec{v}_2 >$$

$$(\lambda_1 - \lambda_2) < \vec{v}_1 , \vec{v}_2 > = 0$$

$$\Rightarrow \quad < \vec{v}_1 , \vec{v}_2 > = 0.$$

Example: $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

Find 3 orthonormal eigenvectors of A.

Sol: $|A - \lambda I| = \begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix}$

$$= -\lambda(\lambda^2 - 1) - (-\lambda - 1) + (1 + \lambda)$$

$$= -\lambda(\lambda+1)(\lambda-1) + (\lambda+1) + (\lambda+1)$$

$$= (\lambda+1)[-\lambda(\lambda-1) + 1 + 1]$$

$$= (\lambda+1)[-\lambda^2 + \lambda + 2]$$

$$= -(\lambda+1)^2(\lambda-2)$$

$\lambda_1 = -1$, $\lambda_2 = 2$

① Plug in $\lambda_1 = -1$ into $(A - \lambda I)\vec{v} = \vec{0}$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\left.\begin{array}{l} v_2 = s \\ v_3 = t \end{array}\right\} \Rightarrow v_1 = -s-t \Rightarrow \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} -s-t \\ s \\ t \end{pmatrix}$

$= s\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$\Rightarrow$ Eigen-Space is Span $\left\{ \overset{\vec{u}_1}{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}, \overset{\vec{u}_2}{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}} \right\}$

But two basis eigen-vectors are not orthogonal.

Apply Gram-Schmidt Procedure:

$$\vec{v}_1 = \vec{u}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{v}_1, \vec{u}_2 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1$$

$$= \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

Verify $A\vec{v}_2 = (-1)\vec{v}_2$

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -1 \end{bmatrix}$$

② Plug in $\lambda = 2$ into $(A - \lambda I)\vec{v} = \vec{0}$

to find $\vec{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

So we get orthogonal eigenvectors

$$\vec{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

and orthonormal eigenvectors:

$$\vec{w}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{pmatrix}$$

$$\vec{w}_2 = \frac{1}{\sqrt{\frac{3}{2}}} \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}\sqrt{\frac{2}{3}} \\ -\frac{1}{2}\sqrt{\frac{2}{3}} \\ \sqrt{\frac{2}{3}} \end{pmatrix}$$

$$\vec{w}_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{1}{3}} \\ \sqrt{\frac{1}{3}} \\ \sqrt{\frac{1}{3}} \end{pmatrix}$$

Use $V = [\vec{w}_1 \ \ \vec{w}_2 \ \ \vec{w}_3]$ $\quad D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Then $A = VDV^{-1}$

and $V^{-1} = V^T$

$$V^T V = \begin{bmatrix} \text{—} \\ \text{—} \\ \text{—} \end{bmatrix} \begin{bmatrix} | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Theorem If $A \in \mathbb{R}^{n \times n}$ is skew-symmetric ($A^T = -A^T$), then

① A has n purely imaginary eigenvalues including repeated ones

② A has n orthogonal complex eigenvectors, i.e.,

all eigenspaces are orthogonal to one another.

Example: $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $\quad \lambda_1 = i \quad \vec{v}_1 = \begin{pmatrix} -i \\ 1 \end{pmatrix}$

$\lambda_2 = -i \quad \vec{v}_2 = \begin{pmatrix} i \\ 1 \end{pmatrix}$

$\langle \vec{v}_1, \vec{v}_2 \rangle = \overline{\vec{v}_2}^T \vec{v}_1 = [-i \quad 1]\begin{bmatrix} -i \\ 1 \end{bmatrix} = (-i)^2 + 1 = 0$

For two complex vectors $\vec{u}, \vec{v} \in \mathbb{C}^n$, the dot product is defined as

$$\langle \vec{u}, \vec{v} \rangle = \overline{\vec{v}}^T \vec{u} = [\overline{v}_1 \quad \overline{v}_2 \quad \overline{v}_3]\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$= \sum_{i=1}^{n} u_i \overline{v}_i$$

So $\langle \vec{u}, \vec{v} \rangle = \overline{\langle \vec{v}, \vec{u} \rangle}$

We define $\vec{u} \perp \vec{v}$ as $\langle \vec{u}, \vec{v} \rangle = 0$.

$$\langle \vec{v}_1, \vec{v}_2 \rangle = \overline{\vec{v}_2}^T \vec{v}_1 = [-i \quad 1]\begin{bmatrix} -i \\ 1 \end{bmatrix} = (-i)^2 + 1 = 0.$$

Theorem If $A \in \mathbb{R}^{n \times n}$ is normal $(AA^T = A^T A)$

then

① A has n complex eigenvalues including repeated ones

② A has n orthogonal complex eigenvectors, i.e.,

all eigenspaces are orthogonal to one another.

Example: $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ is normal

$$A^TA = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$AA^T = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Ex: True or False

① If A is real symmetric, then A is normal.

② If A is skew-symmetric, then A is normal.

$$A^T = -A \Rightarrow \begin{cases} AA^T = A \cdot (-A) = -A^2 \\ A^TA = (-A) \cdot A = -A^2 \end{cases}$$