- A matrix of size $m \times n$ has $\begin{cases} mn & \text{entries} \\ m & \text{rows} \\ n & \text{cols} \end{cases}$

$\mathbb{R}^{m \times n}$

$$\begin{pmatrix} 1 & 0 & 2 & 4 \\ -1 & 6 & 8 & 4 \\ 0 & 0 & 2 & 1 \end{pmatrix} \text{ 2nd row} \quad 3 \times 4$$

$\hookrightarrow$ 3rd col

- A row vector with n entries is a matrix of size $1 \times n$ $\mathbb{R}^{1 \times n}$

$$\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$$

- A col vector with n enties is a matrix of size $n \times 1$ ($\mathbb{R}^{n \times 1}$) $\mathbb{R}^n$

$$\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

- $\mathbb{R}^{n \times n}$ are square matrices

- Matrix-Vector Mul for $A \in \mathbb{R}^{m \times n}$, $\vec{v} \in \mathbb{R}^n$

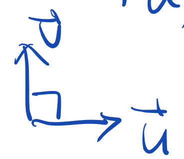
$$\underset{A}{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{pmatrix}} \underset{\vec{v}}{\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}} \underset{①}{=} a\begin{pmatrix} 1 \\ 5 \end{pmatrix} + b\begin{pmatrix} 2 \\ 6 \end{pmatrix} + c\begin{pmatrix} 3 \\ 7 \end{pmatrix} + d\begin{pmatrix} 4 \\ 8 \end{pmatrix}$$

$$= \begin{pmatrix} a \\ 5a \end{pmatrix} + \begin{pmatrix} 2b \\ 6b \end{pmatrix} + \begin{pmatrix} 3c \\ 7c \end{pmatrix} + \begin{pmatrix} 4d \\ 8d \end{pmatrix}$$

$$= \begin{pmatrix} a+2b+3c+4d \\ 5a+6b+7c+8d \end{pmatrix}$$

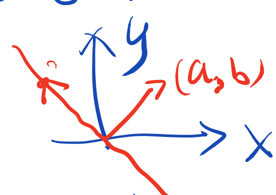
$$\overset{②}{=} \begin{pmatrix} \boxed{1 \quad 2 \quad 3 \quad 4} \\ \boxed{5 \quad 6 \quad 7 \quad 8} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} * \\ * \end{pmatrix}$$

- $\vec{u} \cdot \vec{v} = 0 \iff$



Line & Plane Eqns

- 2D



$$ax + by = c \iff y = -\frac{a}{b}x + \frac{c}{b}$$

Why $ax+by=0$ is a line?

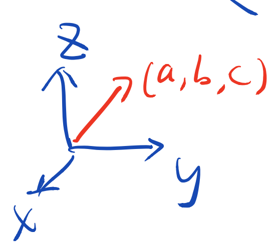
$$\Updownarrow$$

$$y = -\frac{a}{b}x$$

$$\boxed{a}x + \boxed{b}y = 0 \iff \begin{pmatrix} a \\ b \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$\iff \begin{pmatrix} a \\ b \end{pmatrix} \perp \begin{pmatrix} x \\ y \end{pmatrix}$$

- 3D

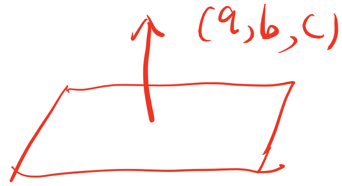


$$\boxed{a}x + \boxed{b}y + \boxed{c}z = 0$$

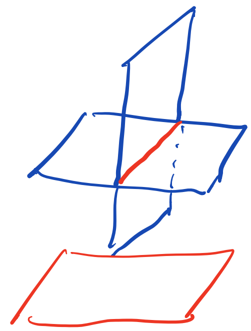
$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$\Rightarrow$ Any vector 90° to $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is a sol to eqn.

$\Rightarrow$ All such vectors form a plane 90° to $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$



- $\begin{cases} x + 2y + 3z = 1 \\ 3x + y + z = 0 \end{cases}$



- $\begin{cases} x + 2y + 3z = 1 \\ 3x + y + z = 0 \\ x + y + z = 0 \end{cases}$

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① Matrix form of linear system
(system of linear eqns)

- $\underbrace{\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}}_{A \text{ Coef Matrix}} \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\vec{x} \text{ Unknow vector}} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \vec{b} \rightarrow$ RHS vector

$[A | \vec{b}]$

- $\begin{pmatrix} x + 2y + 3z \\ 3x + y + z \\ x + y + z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{cases} x + 2y + 3z = 1 \\ 3x + y + z = 0 \\ x + y + z = 0 \end{cases}$

Ex: $\begin{cases} x + 3y + z = 0 \\ -x - y = 1 \end{cases}$

$\Leftrightarrow \begin{cases} x + 3y + z = 0 \\ -x - y + 0 \cdot z = 1 \end{cases} \Leftrightarrow \underbrace{\begin{pmatrix} 1 & 3 & 1 \\ -1 & -1 & 0 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\vec{x}} = \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\vec{b}}$

② $A\vec{x} = \vec{b}$

Augmented Matrix $[A \mid \vec{b}]$

$$[A \mid \vec{b}] = \left(\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ -1 & -1 & 0 & 1 \end{array}\right)$$

③ Row Operations to matrices

Type I Row Op : switch two rows

II : mul a row by a $\underbrace{}_{\text{nonzero}}$ scalar

III : mul a row by a scalar then add it to another row

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 3 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right) \xrightarrow{r1 \leftrightarrow r2} \left(\begin{array}{ccc|c} 3 & 1 & 1 & 0 \\ 1 & 2 & 3 & 1 \\ 1 & 1 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{\frac{1}{3}r1 \rightarrow r1} \left(\begin{array}{ccc|c} 1 & \frac{1}{3} & \frac{1}{3} & 0 \\ 1 & 2 & 3 & 1 \\ 1 & 1 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{\frac{1}{2}r2 \rightarrow r2} \left(\begin{array}{ccc|c} 1 & \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{2} & 1 & 3/2 & 1/2 \\ 1 & 1 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{(-1)\cdot r1 + r3 \rightarrow r3} \left(\begin{array}{ccc|c} 1 & \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{2} & 1 & 3/2 & 1/2 \\ 0 & \frac{2}{3} & \frac{2}{3} & 0 \end{array}\right)$$

④ Gaussian Elimination:

the goal is to transform $[A \mid \vec{b}]$ into this form

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{array}\right]$$

Example: $\begin{cases} x+2y-z=0 \\ 2y-z=1 \\ x+3z=-1 \end{cases}$

Sol: $\begin{cases} x+2y-z=0 \\ 0\cdot x+2y-z=1 \\ 1\cdot x+0\cdot y+3z=-1 \end{cases}$

Step 0: $\begin{bmatrix} 1 & 2 & -1 \\ 0 & 2 & -1 \\ 1 & 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

pivot
leading coef
leading one

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 2 & -1 & 1 \\ \underline{\underline{1}} & 0 & 3 & -1 \end{array}\right]$$

Step 1: Start with first nonzero col,
do Type I op to make (1,1)-entry to be nonzero
do Type II op to make it 1.

Step 2: Do Type III op to elliminate all the other nonzero entries in the same col with pivot.

$$r_3 - r_1 \to r_3 \quad \Longrightarrow \quad \left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 2 & -1 & 1 \\ 0 & -2 & 4 & -1 \end{array}\right)$$

Step 3: do similar things to the submatrix (ignoring the first col)

$$\tfrac{1}{2} r_2 \to r_2 \quad \Longrightarrow \quad \left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} \\ 0 & -2 & 4 & -1 \end{array}\right)$$

$$-2 \cdot r_2 + r_1 \to r_1 \quad \Longrightarrow \quad \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} \\ 0 & -2 & 4 & -1 \end{array}\right)$$

$$2 \cdot r_2 + r_3 \to r_3 \quad \Longrightarrow \quad \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 3 & 0 \end{array}\right)$$

$$\xrightarrow{\frac{1}{3}r_3 \to r_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{\frac{1}{2}\cdot r_3 + r_2 \to r_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$$\begin{cases} 1\cdot x + 0\cdot y + 0 z = -1 \\ 0\cdot x + 1\cdot y + 0\cdot z = \frac{1}{2} \\ 0\cdot x + 0\cdot y + 1\cdot z = 0 \end{cases}$$

Def Row Echelon Form (REF)

1. zero rows are at bottom
2. leading coef of a nonzero row

   pivot

   first nonzero entry in a row

   is always strictly to the right of the pivot of the row above it.

Ex:
$$\begin{pmatrix} 1 & a & b & c & 0 \\ 0 & 0 & 2 & d & f \\ 0 & 0 & 0 & 1 & g \end{pmatrix}_{3\times 5}$$

REF ✓

$$\begin{pmatrix} 1 & a & b & c & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & d & f \end{pmatrix}$$