The following $\boxed{\text{real}}$ matrices are always diagonalizable with orthogonal/orthonormal eigenvectors

① Real Symmetric Matrix $A = A^T$

$A = A^T \Rightarrow$ all eigenvalues are real $\Rightarrow$ real orthogonal eigenvectors

② Real Skew-Symmetric $A^T = -A$

$A = A^T \Rightarrow$ all eigenvalues are imaginary $\Rightarrow$ complex eigenvectors

③ Normal Matrix $AA^T = A^TA$

Def: If $\vec{x}, \vec{y} \in \mathbb{C}^n$, then the dot product is

$$\langle \vec{x}, \vec{y} \rangle = \vec{y}^* \vec{x} = \overline{\vec{y}}^T \vec{x}$$

$*$ denotes conjugate transpose

Ex: $$\left\langle \begin{bmatrix} 1+i \\ 1-i \end{bmatrix}, \begin{bmatrix} i \\ 1 \end{bmatrix} \right\rangle = \begin{bmatrix} \bar{i} & \bar{1} \end{bmatrix} \begin{bmatrix} 1+i \\ 1-i \end{bmatrix}$$

$$= \begin{bmatrix} -i & 1 \end{bmatrix} \begin{bmatrix} 1+i \\ 1-i \end{bmatrix}$$

$$= -i(1+i) + 1 \cdot (1-i)$$

$$= -i + 1 + 1 - i = 2 - 2i$$

Ex: $B = \begin{bmatrix} i & -i \\ -i & i \end{bmatrix}$ is not real symmetric

But still diagonalizable: $$B = i \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= i V D V^{-1}$$

$$= V (iD) V^{-1}$$

Theorem: If $A \in \mathbb{C}^{n \times n}$ satisfies $A^* A = A A^*$ (normal),
$\bar{A}^T A = A \bar{A}^T$

then A is diagonalizable with orthonormal eigenvectors.

An example of application :

The heat equation is given as

$$\frac{\partial}{\partial t} u(x,t) = c \frac{\partial^2}{\partial x^2} u(x,t)$$

or $\quad u_t = c\, u_{xx}$

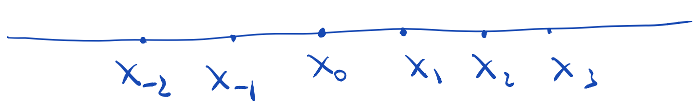
where c is a constant for thermal diffusivity of the metal bar.

An initial value problem is $\begin{cases} u_t = c\, u_{xx} \\ u(x,0) = f(x) \end{cases}$

for a given initial temperature $f(x)$.

To approximate the solution, we consider grid points x_i with uniform spacing Δx

$$x_i = i \Delta x$$



Let $u_i(t) = u(x_i, t)$, denoted by u_i.

The second order derivative can be approximated by

$$u_{xx}(x_i, t) \approx \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2}$$

because

$$u(x_{i+1}) = u(x_i + \Delta x) \approx u(x_i) + \Delta x\, u_x(x_i) + \frac{\Delta x^2}{2} u_{xx}(x_i) + O(\Delta x^3)$$

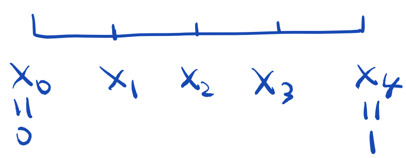
$$u(x_{i-1}) = u(x_i - \Delta x) \approx u(x_i) - \Delta x\, u_x(x_i) + \frac{\Delta x^2}{2} u_{xx}(x_i) + O(\Delta x^3)$$

$$\Rightarrow \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2} \approx u_{xx}(x_i) + O(\Delta x)$$

Suppose we consider a finite interval $x \in [0, 1]$

and impose boundary condition $u(0,t) = u(1,t) = 0$

which means the bar is cold on two ends.



$$\Delta x = \frac{1}{4}$$

$$x_i = \Delta x \cdot i = \frac{i}{4}$$

$$u_0(t) = 0 \qquad\qquad u_t = c\, u_{xx}$$

$$u_4(t) = 0$$

$$\frac{d}{dt} u_1(t) = c\frac{u_0 - 2u_1 + u_2}{\Delta x^2} = c\,\frac{-2u_1 + u_2}{\Delta x^2}$$

$$\frac{d}{dt} u_2(t) = c\frac{u_1 - 2u_2 + u_3}{\Delta x^2}$$

$$\frac{d}{dt} u_3(t) = c\frac{u_2 - 2u_3 + u_4}{\Delta x^2} = c\,\frac{u_2 - 2u_3}{\Delta x^2}$$

$$\Rightarrow \frac{d}{dt}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \frac{c}{\Delta x^2}\begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

$$\left.\begin{array}{l}\text{More points}\\ \text{Smaller } \Delta x\end{array}\right\} \Rightarrow \left\{\begin{array}{l}\cdot \text{ Better approximation}\\ \cdot \text{ More equations}\\ \cdot \text{ Larger system/matrix}\end{array}\right.$$

$$\frac{d}{dt}\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{pmatrix} = \frac{c}{\Delta x^2}\begin{pmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{pmatrix}$$

Question:

What are the eigenvalue/eigenvector of

$$A = \frac{1}{\Delta x^2}\begin{pmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{pmatrix} ?$$

$A\vec{u}$ is an approximation to $\frac{\partial^2}{\partial x^2} u(x)$.

So $A\vec{u} = a\,\vec{u}$ is similar to

$$\frac{\partial^2}{\partial x^2} u(x) = a\, u(x)$$

a & $u(x)$ are eigenvalue and eigenfunction for the operator $\frac{\partial^2}{\partial x^2}$

Q: how to find $u''(x) = a u(x)$?

$$\frac{d}{dx}\begin{pmatrix} u(x) \\ u'(x) \end{pmatrix} = \begin{pmatrix} u' \\ u'' \end{pmatrix} = \begin{pmatrix} u' \\ au \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ a & 0 \end{pmatrix}\begin{pmatrix} u \\ u' \end{pmatrix}$$

$$\det\left[\begin{pmatrix} 0 & 1 \\ a & 0 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]$$

$$= \begin{vmatrix} -\lambda & 1 \\ a & -\lambda \end{vmatrix} = \lambda^2 - a \Rightarrow \lambda = \pm\sqrt{a}$$

① If $a \geq 0$, let $b = \sqrt{a}$, $\lambda = \pm b$

$\mathrm{Span}\{e^{bt}\vec{v}_1, e^{-bt}\vec{v}_2\}$ is the sol set

② If $a < 0$, let $b = \sqrt{-a}$, $\lambda = \pm ib$

$\mathrm{Span}\{e^{ibt}\vec{v}_1, e^{-ibt}\vec{v}_2\}$ is the sol set.

$e^{i\theta} = \cos\theta + i\sin\theta$

$\Rightarrow$ All possible functions satisfying

$u''(x) = a\,u(x)$ for some $a \in \mathbb{C}$

are e^{bx}, $\cos(bx)$, $\sin(bx)$, $bx+c$

The original problem is $\begin{cases} u_t = c\, u_{xx}, & x \in (0,1) \\ u(0,t) = u(1,t) = 0 \end{cases}$

So $A\vec{u} = a\vec{u}$ is an approximation to

$$\frac{\partial^2}{\partial x^2} u(x) = a\, u(x) \quad \text{with} \quad u(0) = u(1) = 0$$

The eigenfunction satisfying $u(0) = u(1) = 0$ can only be $\sin(n\pi x)$, n is integer.

Remark: The eigenvectors of

$$A = \frac{1}{\Delta x^2}\begin{pmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{pmatrix}$$

happen to consist of point values of $\sin(n\pi x)$.