- (Abstract) Vector Space
- Invertible Matrix
- Solution to $A\vec{x} = \vec{b}$
- ODE
- Diagonalization
- SVD

- Examples of (abstract) vector spaces:

  1) $\mathbb{R}^n$ 2) $\mathbb{R}^{1\times m}$ 3) $\mathbb{R}^{m\times n}$ 4) $P_2(\mathbb{R})$ quadratic polynomial

  5) $V = \{ f(x) : f'(x) \text{ and } f''(x) \text{ exist for any } x \}$

  6) $W = \{ \text{solutions to } y''(t) - y'(t) + y(t) = 0 \}$

  I. two operations are defined

  II. closed under two operations

  ($\Rightarrow$ zero "vector" is included)

- A basis of a vector space is a linearly independent set of vectors which can span the vector space
- The number of basis vectors is called the dimension.

  If a vector space V has dimension n, then

  1) The maximum number of independent vectors is n

  2) The minimum number of vectors spanning V is n

- Example: $V = \{\text{all real symmetric } 3\times 3 \text{ matrices}\}$

$$= \left\{ \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} : \forall\, a, b, c, d, e, f \right\}$$

$\dim(V) = 6.$

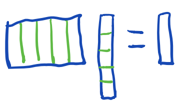
A basis is $\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\}$

- $A \in \mathbb{R}^{m\times n}$

$\mathrm{Null}(A) = \{ \vec{x} \in \mathbb{R}^n : A\vec{x} = \vec{0} \}$

$\mathrm{Rank}(A) + \mathrm{Nullity}(A) = n$

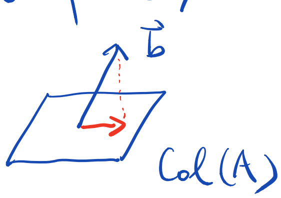
$A\vec{x} = \vec{b}$ has solutions $\iff \vec{b} \in \mathrm{Col}(A)$



If $\vec{b} \in \mathrm{Col}(A)$, then $A\vec{x} = \vec{b}$ has at least one sol $\vec{x}_p$.

The sol set can be written as $\vec{x}_p + \mathrm{Null}(A)$

- What if $\vec{b} \notin \mathrm{Col}(A)$?



$A \in \mathbb{R}^{m\times n}$, $m > n$

$A \quad \vec{x} = \vec{b}$

$A\vec{x} = \vec{b}$

$A^TA\hat{x} = A^T\vec{b}$

Solve for $\hat{x}$ $\left[= (A^TA)^{-1}A^T\vec{b} \right]$

The projection of $\vec{b}$ is

$A\hat{x} = A(A^TA)^{-1}A^T\vec{b}$

Col(A) is at most n-dim

$\vec{b}$ is in m-dim space.

① In order for $(A^TA)^{-1}$ to exist, A need to have independent column vectors.

If we want to project a vector $\vec{b}$ onto a subspace or a span, find a basis for that subspace. Use basis column vectors to form A.

② If $A = [\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3]$ has orthogonal columns,

then $A^TA =$

$$= \begin{bmatrix} \|\vec{v}_1\|^2 & 0 & 0 \\ 0 & \|\vec{v}_2\|^2 & 0 \\ 0 & 0 & \|\vec{v}_3\|^2 \end{bmatrix}$$

$\Rightarrow$ The projection of $\vec{b}$ onto $\mathrm{Col}(A)$ is also

$$[A(A^TA)^{-1}A^T\vec{b} =] \quad \frac{\langle \vec{b}, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 + \frac{\langle \vec{b}, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 + \frac{\langle \vec{b}, \vec{v}_3 \rangle}{\|\vec{v}_3\|^2}\vec{v}_3$$

Example: Projection of $\vec{b}$ onto $\mathrm{Span}\{\vec{v}\}$ is $\dfrac{\langle \vec{b}, \vec{v} \rangle}{\|\vec{v}\|^2}\vec{v}$

③ If $A = [\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3]$ has orthonormal columns,

then $A^TA =$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow$ The projection of $\vec{b}$ onto $\mathrm{Col}(A)$ is also

$$[A(A^TA)^{-1}A^T\vec{b} =] \quad \langle \vec{b}, \vec{v}_1 \rangle \vec{v}_1 + \langle \vec{b}, \vec{v}_2 \rangle \vec{v}_2 + \langle \vec{b}, \vec{v}_3 \rangle \vec{v}_3$$

- Gram-Schmit for generating orthogonal vectors

Question: Given $\mathrm{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \cdots, \vec{v}_7\}$

how to find an orthogonal basis?

Answer: 1) Find a basis, say, $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

2) Apply Gram-Schmit to $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

$$\vec{u}_1 = \vec{v}_1$$

$$\vec{u}_2 = \vec{v}_2 - \frac{\langle \vec{v}_2, \vec{u}_1 \rangle}{\|\vec{u}_1\|^2} \vec{u}_1$$

$$\vec{u}_3 = \vec{v}_3 - \frac{\langle \vec{v}_3, \vec{u}_1 \rangle}{\|\vec{u}_1\|^2} \vec{u}_1 - \frac{\langle \vec{v}_3, \vec{u}_2 \rangle}{\|\vec{u}_2\|^2} \vec{u}_2$$

- When is A invertible?

The following are equivalent for $A \in \mathbb{R}^{n\times n}$

① A^{-1} exists

② rank(A) = n.

③ Nullity(A) = 0 ($\Leftrightarrow$ Rank(A) = n)

④ $A\vec{x} = \vec{0}$ has only zero sol.

⑤ $A\vec{x} = \vec{b}$ has a unique sol $A^{-1}\vec{b}$.

⑥ row space of A has dim n

⑦ col — — — —

⑧ $A = E_1 \cdots E_n$ a product of elemetary matrices

⑨ det(A) $\neq 0$

dim Theorem:

Nullity + Rank = # of cols

General eigen-decomposition $A = VJV^{-1}$
where J is upper triangular with λ_i on diagonal.

$$\det(A) = \det(VJV^{-1}) = \det(JV^{-1}V) = \det(J)$$
$$= \lambda_1 \lambda_2 \cdots \lambda_n$$

⑩ All n complex eigenvalues are nonzero

⑪ All singular values are positive.

- When is $A \in \mathbb{R}^{n \times n}$ diagonalizable?

① n independent eigenvectors

② $AA^T = A^TA$ $\quad A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

③ $A^T = A$

④ $A^T = -A$

- Diagonalization $A = VDV^{-1}$.

① $A = VDV^{-1} \Leftrightarrow A\vec{v}_i = d_i \vec{v}_i$

② real orthonormal eigenvectors $\vec{v}_i \Leftrightarrow A = VDV^T$

$$\Uparrow \qquad\qquad\qquad\qquad \Downarrow$$
$$A = A^T \qquad\qquad\qquad\qquad A = A^T$$

- ODE: $\begin{cases} \dfrac{d}{dt}\vec{u}(t) = A\vec{u}(t) \\ \vec{u}(0) = \vec{u}_0 \end{cases}$

$$\Rightarrow \vec{u}(t) = e^{At}\vec{u}(0)$$

$$A = VDV^{-1} \Rightarrow e^{At} = Ve^{Dt}V^{-1}\vec{u}(0)$$

$$\Rightarrow \vec{u}(t) = V e^{Dt} \underbrace{V^{-1}\vec{u}(0)}_{\vec{x}} \qquad V\vec{x} = \vec{u}(0)$$

- SVD : $A = U\Sigma V^T$, $A \in \mathbb{R}^{n\times n}$

① $\Sigma = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix}$, $\sigma_i \geq 0$

$\sigma(A) = \sqrt{\lambda(A^TA)} = \sqrt{\lambda(AA^T)}$

② U has orthonormal cols

$AA^T = U\Sigma U^T \Rightarrow \vec{u}_i$ are eigenvectors of AA^T

③ V has orthonormal cols

$A^TA = V\Sigma V^T \Rightarrow \vec{v}_i$ are eigenvectors of A^TA

④ $A\vec{v}_i = \sigma_i \vec{u}_i$

Example:

$$\begin{vmatrix} 0 & 0 & 1 & 0 \\ 2 & -1 & 4 & 3 \\ 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 3 \end{vmatrix} = 1\cdot(-1)^{1+3}\cdot\begin{vmatrix} 2 & -1 & 3 \\ 1 & 0 & 1 \\ 0 & 2 & 3 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & -1 & 3 \\ 1 & 0 & 1 \\ 0 & 2 & 3 \end{vmatrix}$$

$$= 2\cdot(-1)^{1+1}\cdot\begin{vmatrix} 0 & 1 \\ 2 & 3 \end{vmatrix}$$

$$+ 1\cdot(-1)^{2+1}\cdot\begin{vmatrix} -1 & 3 \\ 2 & 3 \end{vmatrix}$$

$$= 2\begin{vmatrix} 0 & 1 \\ 2 & 3 \end{vmatrix} - \begin{vmatrix} -1 & 3 \\ 2 & 3 \end{vmatrix}$$

$$= 2\cdot(0-2) - (-3-6)$$