

A few concepts for matrices.

Let $c \in \mathbb{R}$ be a scalar
| $A \in \mathbb{R}^{m \times n}$ be a matrix \rightarrow then define

① Scalar multiplication to a matrix:

cA is defined as multiplying c to each entry.

$$\text{Example: } 2 \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix}$$

② Matrix Addition: $A, B \in \mathbb{R}^{m \times n}$

$A+B$ is defined as addition for each entry.

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 6 & 8 \\ 10 & 12 \end{pmatrix}$$

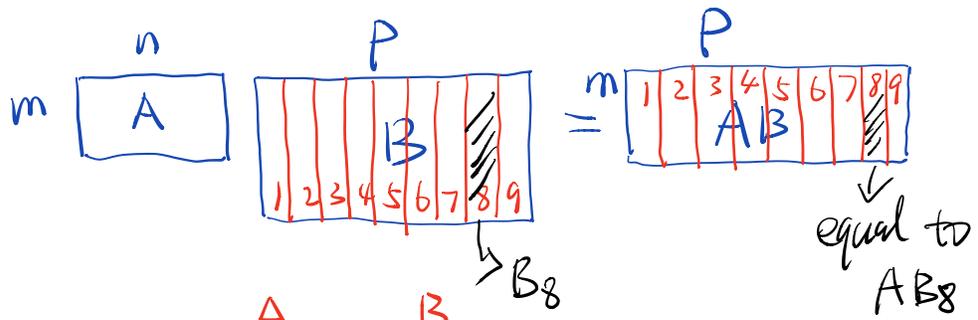
③ Matrix-Matrix Multiplication: $\begin{cases} A \in \mathbb{R}^{m \times n} \\ B \in \mathbb{R}^{n \times p} \end{cases}$

AB is called product of A and B .

1) $AB \in \mathbb{R}^{m \times p}$

$$\begin{matrix} m & n & & p \\ \boxed{A} & & \boxed{B} & \\ m \times n & & n \times p & \\ & & & = m \times p \\ & & & \boxed{AB} \end{matrix}$$

2) Let B_j ($j=1, \dots, p$) be cols of B ,
then j -th col of AB is equal to AB_j



Example:

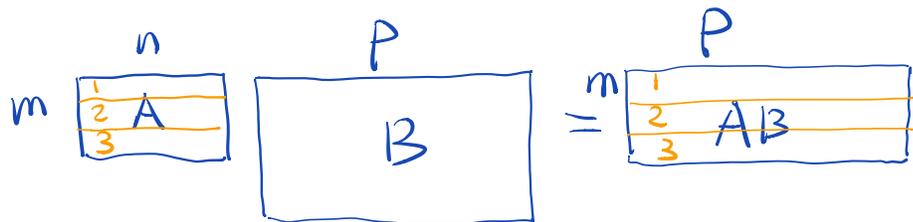
$$\begin{pmatrix} 1 & 2 \\ -7 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ -7 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ -7 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 7 \\ -7 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ -7 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 7 \\ 0 & -7 \end{pmatrix}$$

2) Let A_i ($i = 1, \dots, m$) be rows of A , then i -th row of AB is equal to $A_i B$



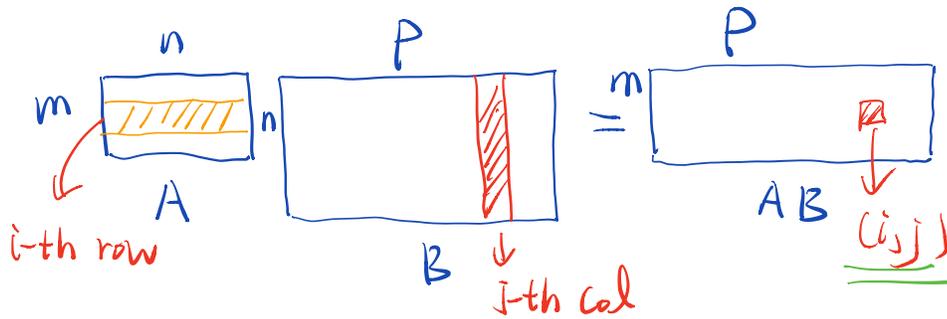
Ex:

$$\begin{pmatrix} 1 & 2 \\ -7 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 7 \end{pmatrix}$$

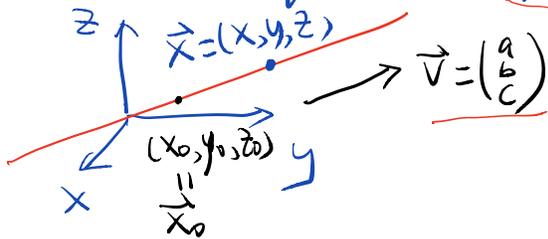
$$\underline{(-1 \ 0)}, \left(\begin{array}{c} 0 \\ 1 \end{array} \right), \left(\begin{array}{c} 1 \\ 3 \end{array} \right) = (0 \ -1)$$

3) The (i, j) -entry of AB is equal to the dot product of A_i and B_j



Ex: $\begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} * & * \\ * & * \end{pmatrix}$

The Line Equation in point-direction form



A line passes the point $\vec{x}_0 = (x_0, y_0, z_0)$ and is parallel to $\vec{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

Let $\vec{x} = (x, y, z)$ be any point on this line.



The vector from \vec{x}_0 to \vec{x} is obtained by

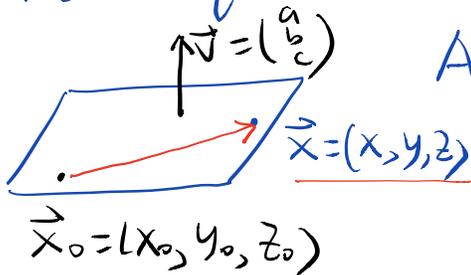
$$\vec{x} - \vec{x}_0 = \begin{pmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{pmatrix}, \text{ which is parallel to } \vec{v}.$$

$$\Rightarrow \vec{x} - \vec{x}_0 = t\vec{v} \text{ for some } t \in \mathbb{R}.$$

$$\Rightarrow \begin{pmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{pmatrix} = t \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + t \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \quad t \in \mathbb{R}.$$

Plane Equation in 3D



A plane $\left\{ \begin{array}{l} \text{passes a point } \vec{x}_0 \\ \perp \vec{v} \end{array} \right.$

$$\Rightarrow \text{The vector } (\vec{x} - \vec{x}_0) \perp \vec{v}$$

$$\Rightarrow \begin{pmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$$

$$\Rightarrow a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$\Rightarrow ax + by + cz = ax_0 + by_0 + cz_0$$

We usually write it as $ax + by + cz = d$.

Linear System

$$\begin{cases} 2x + 4y - 2z = 2 \\ 4x + 9y - 3z = 8 \\ -2x - 3y + 7z = 10 \end{cases}$$

Matrix Form

$$\begin{pmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \\ 10 \end{pmatrix}$$

Augmented Matrix

$$\left(\begin{array}{ccc|c} 2 & 4 & -2 & 2 \\ 4 & 9 & -3 & 8 \\ -2 & -3 & 7 & 10 \end{array} \right)$$

Gaussian Elimination: use row ops to get either REF or RREF

Row Echelon Form
(REF)

- ① zero rows are at bottom
- ② position of leading coeffs

first nonzero entry in each row

1) $\left(\begin{array}{ccc|c} \textcircled{1} & 0 & 0 & 4 \\ 0 & \textcircled{2} & 0 & 5 \\ 0 & 0 & \textcircled{3} & 6 \end{array} \right)$ ✓

2) $\left(\begin{array}{ccc|c} \textcircled{1} & 1 & 0 & 0 \\ 0 & 0 & \textcircled{2} & 1 \\ 0 & 0 & 0 & 0 \end{array} \right)$ ✓

4) $\left(\begin{array}{ccc|c} \textcircled{1} & 0 & 0 & 0 \\ 0 & 0 & \textcircled{1} & 0 \\ 0 & 0 & \textcircled{1} & 1 \end{array} \right)$ ✗

3) $\left(\begin{array}{ccc|c} \textcircled{1} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \textcircled{1} & 0 & 0 \end{array} \right)$ ✗

5) $\left(\begin{array}{ccc|c} \textcircled{2} & 0 & 0 & 1 \\ \textcircled{2} & 2 & 2 & 2 \\ 0 & 0 & \textcircled{2} & 3 \end{array} \right)$ ✗

Reduced Row Echelon Form

RREF

- ① already a REF
- ② all pivots are ones
- ③ any col containing leading ones has only one nonzero entry.

$$1) \begin{pmatrix} 1 & 2 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & | & 1 \end{pmatrix} \checkmark \quad 3) \begin{pmatrix} 1 & 0 & 0 & | & 1 \\ 0 & 0 & 0 & | & 2 \\ 0 & 0 & 1 & | & 3 \end{pmatrix} \times$$

$$2) \begin{pmatrix} 1 & 0 & 0 & | & a \\ 0 & 1 & 0 & | & b \\ 0 & 0 & 1 & | & c \end{pmatrix} \checkmark \quad 4) \begin{pmatrix} 2 & 0 & 0 & | & 0 \\ 0 & 2 & 0 & | & 0 \\ 0 & 0 & 2 & | & 0 \end{pmatrix} \times$$

Ex: Matrix Form

$$\begin{pmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \\ 10 \end{pmatrix}$$

Augmented Matrix

$$\left(\begin{array}{ccc|c} 2 & 4 & -2 & 2 \\ 4 & 9 & -3 & 8 \\ -2 & -3 & 7 & 10 \end{array} \right)$$

How to solve it by Gaussian Elimination:

$$\left(\begin{array}{ccc|c} 2 & 4 & -2 & 2 \\ 4 & 9 & -3 & 8 \\ -2 & -3 & 7 & 10 \end{array} \right)$$

$$\frac{1}{2}r_1 \rightarrow r_1 \Rightarrow \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 4 & 9 & -3 & 8 \\ -2 & -3 & 7 & 10 \end{array} \right)$$

$$-4r_1 + r_2 \rightarrow r_2 \Rightarrow \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & 1 & 4 \\ -2 & -3 & 7 & 10 \end{array} \right)$$

$$2r_1 + r_3 \rightarrow r_3$$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 1 & 5 & 12 \end{array} \right)$$

$$r_3 - r_2 \rightarrow r_3$$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 4 & 8 \end{array} \right)$$

$$r_1 - 2r_2 \rightarrow r_1$$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & -3 & -7 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 4 & 8 \end{array} \right)$$

$$\frac{1}{4}r_3 \rightarrow r_3$$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & -3 & -7 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 1 & 2 \end{array} \right)$$

either

① Solve it by substitution backwards:

or RREF $z=2 \Rightarrow y=2 \Rightarrow x=-1$

$$\textcircled{2} r_2 - r_3 \rightarrow r_2$$

$$\begin{array}{l} \xrightarrow{\hspace{2cm}} \\ 3r_3 + r_1 \rightarrow r_1 \end{array} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{array} \right)$$

Different Scenarios of RREF or REF

$$\textcircled{1} \left(\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{array} \right) \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\textcircled{2} \begin{pmatrix} \textcircled{1} & 0 & 0 & | & a \\ 0 & \textcircled{1} & 0 & | & b \\ 0 & 0 & 0 & | & \textcircled{1} \end{pmatrix} \Rightarrow \text{contradiction} \Rightarrow \text{no sol at all.}$$

$$0 \cdot x + 0 \cdot y + 0 \cdot z = 1$$

$$\textcircled{3} \begin{pmatrix} \textcircled{1} & 2 & 3 & | & 0 \\ 0 & \textcircled{1} & -1 & | & 1 \\ 0 & 0 & 0 & | & 0 \end{pmatrix} \text{ The cols without pivots}$$

correspond to free parameters.

\swarrow
 \searrow
 z is free

Set $z = t$, then solve for the others.

$$\text{Second row} \Rightarrow y - z = 1 \Rightarrow y = 1 + z = 1 + t$$

$$\text{First row} \Rightarrow x + 2y + 3z = 0$$

$$\begin{aligned} \Rightarrow x &= -2y - 3z \\ &= -2(1+t) - 3t \\ &= -5t - 2 \end{aligned}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -5t - 2 \\ 1 + t \\ t \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -5t \\ t \\ t \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -5 \\ 1 \\ 1 \end{pmatrix},$$

$$\forall t \in \mathbb{R}.$$

$$\textcircled{4} \begin{pmatrix} \textcircled{1} & \textcircled{1} & \textcircled{1} & | & 2 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix} \quad x + y + z = 2$$

$$[\textcircled{1} \quad \boxed{1} \quad \boxed{1} \quad | \quad 2]$$

$$\text{Set } \begin{cases} y = s \\ z = t \end{cases}$$

$$x = 2 - y - z = 2 - s - t.$$

$$\begin{aligned}
 \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \begin{pmatrix} 2-s-t \\ s \\ t \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -s \\ s \\ 0 \end{pmatrix} + \begin{pmatrix} -t \\ 0 \\ t \end{pmatrix} \\
 &= \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \forall s, t \in \mathbb{R}.
 \end{aligned}$$

Ex: The equation $x+y+z+t=0$ represents
 a 3D plane in x - y - z - t space. Why?

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \end{array} \right]$$

Ex:
$$\begin{cases} x+y+z+t=2 \\ 2x-y+t=3 \end{cases}$$

Matrix Form
$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & | & 2 \\ 2 & -1 & 0 & 1 & | & 3 \end{bmatrix}$$

$$\xrightarrow{-2r_1 + r_2 \rightarrow r_2} \begin{bmatrix} 1 & 1 & 1 & 1 & | & 2 \\ 0 & -3 & -2 & -1 & | & -1 \end{bmatrix}$$

$$\text{Set } \begin{cases} z = u \\ t = v \end{cases}$$

$$\begin{aligned} -3y - 2z - t &= -1 \Rightarrow y = -\frac{2}{3}z - \frac{1}{3}t + \frac{1}{3} \\ &= -\frac{2}{3}u - \frac{1}{3}v + \frac{1}{3} \end{aligned}$$

$$x + y + z + t = 2$$

$$\Rightarrow x = 2 - y - z - t$$

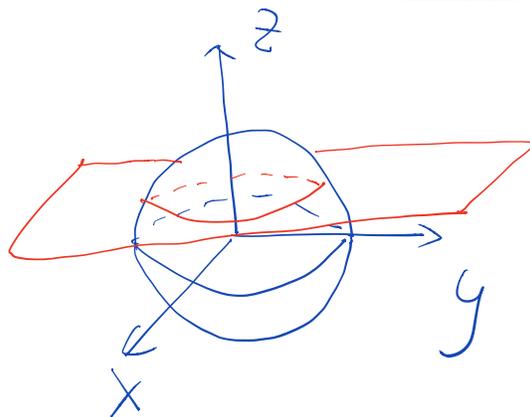
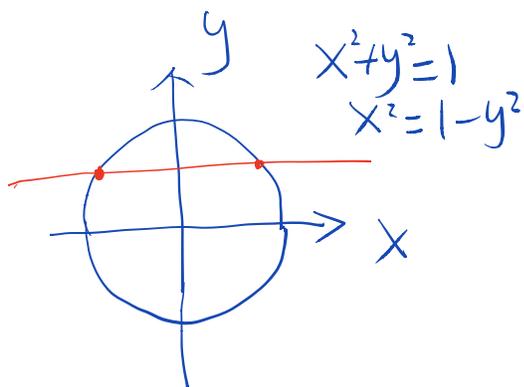
$$= 2 + \frac{2}{3}u + \frac{1}{3}v - \frac{1}{3} - u - v = -\frac{1}{3}u - \frac{2}{3}v + \frac{5}{3}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} -\frac{1}{3}u - \frac{2}{3}v + \frac{5}{3} \\ -\frac{2}{3}u - \frac{1}{3}v + \frac{1}{3} \\ u \\ v \end{pmatrix}$$

$$= \begin{pmatrix} 5/3 \\ 1/3 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -1/3 u \\ -2/3 u \\ u \\ 0 \end{pmatrix} + \begin{pmatrix} -2/3 v \\ -1/3 v \\ 0 \\ v \end{pmatrix}$$

$$= \begin{pmatrix} 5/3 \\ 1/3 \\ 0 \\ 0 \end{pmatrix} + u \begin{pmatrix} -1/3 \\ -2/3 \\ 1 \\ 0 \end{pmatrix} + v \begin{pmatrix} -2/3 \\ -1/3 \\ 0 \\ 1 \end{pmatrix}$$

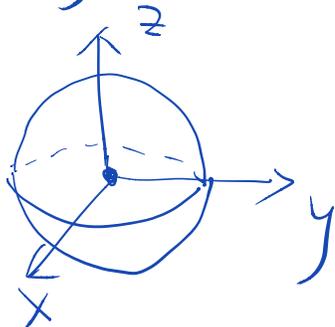
$\forall u, v \in \mathbb{R}.$



$$x^2 + y^2 + z^2 + t^2 = 1$$

$$x^2 + y^2 + z^2 = 1 - t^2$$

$$-1 \leq t \leq 1$$



$t = 1$: origin

$t = 0.9$: sphere

$t = 0.5$: larger sphere

$t = 0$: largest sphere

$t = 0.5 = \text{smaller}$

$t = 1 : \text{origin}$