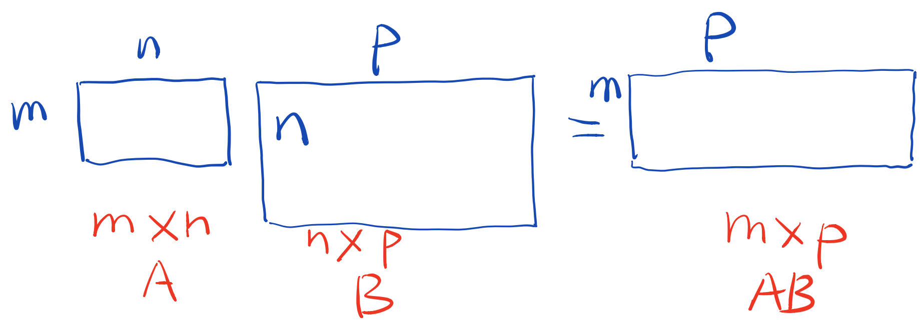
- Matrix – Matrix Multiplication: $\left.\begin{array}{l} A \in \mathbb{R}^{m\times n} \\ B \in \mathbb{R}^{n\times p}\end{array}\right| \Rightarrow AB \in \mathbb{R}^{m\times p}$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ -1 & 1 & 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 1 & -1 \\ 2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 7 & 5 & 2 \\ 2 & 1 & 1 \end{pmatrix}$$



- Rules of Matrix–Matrix Multiplication:

① For $A, B \in \mathbb{R}^{n\times n}$, usually $AB \neq BA$

$$\begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 7 \\ 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} & 0 \\ & \end{pmatrix}$$

② For three matrices A, B, C, if ABC is well-define (sizes match), then

$$(AB)C = A(BC)$$

Example: $\begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

$$\begin{pmatrix}1&2\\-1&0\end{pmatrix}\begin{pmatrix}0&1\\1&3\end{pmatrix}\begin{pmatrix}1\\3\end{pmatrix}$$

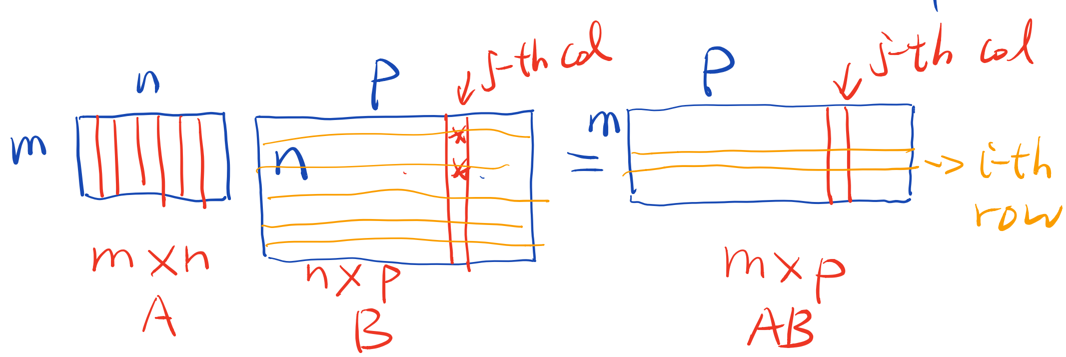
③ $A(B+C) = AB + AC$

$(A+B)C = AC + BC$

- Definition: Identity matrix is a square matrix with $\begin{cases} a_{ii} = 1, & \forall i \\ a_{ij} = 0, & \text{if } i \neq j \end{cases}$

$$\underset{\substack{I_{2\times2} \\ \text{or } I_2}}{\begin{pmatrix}1&0\\0&1\end{pmatrix}}, \quad \underset{\substack{I_{3\times3} \\ I_3}}{\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}}, \quad \underset{\substack{I_{4\times4} \\ I_4}}{\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}}$$

- The j-th col in AB is a linear combination of all cols of A with j-th col of B as coef



$$\begin{pmatrix}1&2\\-1&0\end{pmatrix}\begin{pmatrix}0&1\\1&3\end{pmatrix} = \begin{pmatrix}2&7\\0&-1\end{pmatrix}$$

- The i-th row in AB is a linear combination of all rows of B with i-th row of A as coef

- $\forall A \in \mathbb{R}^{n\times n}$, for the identity matrix I of the same size, we have

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$$

$$AI = A$$
$$IA = A$$

- <u>Definition</u>: $A \in \mathbb{R}^{n\times n}$ is invertible if there is a matrix $B \in \mathbb{R}^{n\times n}$ s.t.

$$AB = I \text{ and } BA = I.$$

If A is invertible, B is called the inverse matrix of A, usually denoted as A^{-1}.

$$AA^{-1} = I \text{ and } A^{-1}A = I$$

Example: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Remark: ① it can be proven that $AB=I \Rightarrow BA=I$ and $BA=I \Rightarrow AB=I$.

So only need to check $AB=I$ (or $BA=I$)

② A^{-1} is unique.

- Linear System

$$\begin{cases} 2x + 4y - 2z = 2 \\ 4x + 9y - 3z = 8 \\ -2x - 3y + 7z = 10 \end{cases}$$

Matrix Form $A\vec{x}=\vec{b}$

$$\begin{pmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 2 \\ 8 \\ 10 \end{pmatrix}$$

Augmented Matrix $[A|\vec{b}]$

$$\left(\begin{array}{ccc|c} 2 & 4 & -2 & 2 \\ 4 & 9 & -3 & 8 \\ -2 & -3 & 7 & 10 \end{array}\right)$$

RREF is

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

$\Rightarrow$ There is a unique sol $\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$.

Facts for an $\boxed{n\times n}$ linear system $A\vec{x}=\vec{b}$

① Number of leading ones in RREF of $[A|\vec{b}]$ is n $\Leftrightarrow$ There is a unique sol.

② If A is invertible, then

$$A\vec{x}=\vec{b}$$

$$\Leftrightarrow A^{-1}A\vec{x}=A^{-1}\vec{b}$$

$$\Leftrightarrow I\vec{x}=A^{-1}\vec{b}$$

$\Leftrightarrow$ $\vec{x}=A^{-1}\vec{b}$ is a sol, and the only sol.

③ If number of leading ones in RREF of $[A|\vec{b}]$ is n, then A is invertible.

In other words, the following are equivalent:

① $A\vec{x}=\vec{b}$ has a unique sol

② REF or RREF of $[A|\vec{b}]$ has n pivots.

③ A is invertible

Example: If $A\vec{x}=\vec{b}$ has many sols (or no sol), then A is not invertible.

- Gaussian Elimination for computing A^{-1}:

$$A = \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix}$$

Step 0: set up $[A \mid I]$

$$\left[\begin{array}{ccc|ccc} 2 & 4 & -2 & 1 & 0 & 0 \\ 4 & 9 & -3 & 0 & 1 & 0 \\ -2 & -3 & 7 & 0 & 0 & 1 \end{array}\right]$$

Step 1: use row ops to transform it to RREF

Step II: ① if we get

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & * & * & * \\ 0 & 1 & 0 & * & * & * \\ 0 & 0 & 1 & * & * & * \end{array}\right]$$

inverse of A

② if we can't get n pivots, then A is not invertible.

Example:

$$\left[\begin{array}{ccc|ccc} 2 & 4 & -2 & 1 & 0 & 0 \\ 4 & 9 & -3 & 0 & 1 & 0 \\ -2 & -3 & 7 & 0 & 0 & 1 \end{array}\right]$$

$(\frac{1}{2}r1 \to r1)$
$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1/2 & 0 & 0 \\ 4 & 9 & -3 & 0 & 1 & 0 \\ -2 & -3 & 7 & 0 & 0 & 1 \end{array}\right]$$

$(-4r1 + r2 \to r2)$
$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1/2 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ -2 & -3 & 7 & 0 & 0 & 1 \end{array}\right]$$

$(2r1 + r3 \to r3)$
$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1/2 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 1 & 5 & 1 & 0 & 1 \end{array}\right]$$

$$(-2r_2+r_1 \to r_1) \quad \left[\begin{array}{ccc|ccc} 1 & 0 & -3 & 9/2 & -2 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 1 & 5 & 1 & 0 & 1 \end{array}\right]$$

$$(-r_2+r_3 \to r_3) \quad \left[\begin{array}{ccc|ccc} 1 & 0 & -3 & 9/2 & -2 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 4 & 3 & -1 & 1 \end{array}\right]$$

$$(\tfrac{1}{4}r_3 \to r_3) \quad \left[\begin{array}{ccc|ccc} 1 & 0 & -3 & 9/2 & -2 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 3/4 & -1/4 & 1/4 \end{array}\right]$$

$$(r_2-r_3 \to r_3) \quad \left[\begin{array}{ccc|ccc} 1 & 0 & -3 & 9/2 & -2 & 0 \\ 0 & 1 & 0 & -11/4 & 5/4 & -1/4 \\ 0 & 0 & 1 & 3/4 & -1/4 & 1/4 \end{array}\right]$$

$$(3r_3+r_1 \to r_1) \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 27/4 & -11/4 & 3/4 \\ 0 & 1 & 0 & -11/4 & 5/4 & -1/4 \\ 0 & 0 & 1 & 3/4 & -1/4 & 1/4 \end{array}\right]$$

$$\Rightarrow A^{-1} = \frac{1}{4}\begin{bmatrix} 27 & -11 & 3 \\ -11 & 5 & -1 \\ 3 & -1 & 1 \end{bmatrix}$$

$$\begin{pmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{pmatrix} \frac{1}{4} \begin{pmatrix} 27 & -11 & 3 \\ -11 & 5 & -1 \\ 3 & -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- Elementary matrices of Type I/II/III are generate by using one row/col. operation of the same type on identity matrix.

Type I: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

Type II: $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Type III: $\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

- The inverse of an Elementary Matrix is generated by opposite operation on identity.

- Let E be an Elementary Matrix.

EA is equivalent to do the row op on A.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} a & b & c \\ g & h & i \\ d & e & f \end{pmatrix}$$

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} 2a & 2b & 2c \\ d & e & f \\ g & h & i \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} a & b & c \\ d+2a & e+2b & f+2c \\ g & h & i \end{pmatrix}$$

AE is equivalent to do the same col op:

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} a & c & b \\ d & f & e \\ g & i & h \end{pmatrix}$$

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a+2b & b & c \\ d+2e & e & f \\ g+2h & h & i \end{pmatrix}$$