## Review

- Matrix-Matrix Multiplication: for $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{n\times p}$, $AB \in \mathbb{R}^{m\times p}$ can be computed or interpreted/viewed as

① $(AB)_{ij}$ is the dot product of i-th row of A and j-th col of B.

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ -1 & 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 1 & -1 \\ 2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 7 & 5 & 2 \\ 2 & 1 & 1 \end{pmatrix}$$

$2\times 4$ $\quad$ $4\times 3$ $\quad$ $2\times 3$

② Each col in AB is obtained by multiplying A to each col of B

③ Each row in AB is obtained by multiplying each row of A to B

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ -1 & 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 1 & -1 \\ 2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 7 & 5 & 2 \\ 2 & 1 & 1 \end{pmatrix}$$

④ Each col in AB is a linear combination of cols in A.

⑤ Each row in AB is a linear combination of rows in B.

Ex: $\begin{pmatrix} 2 \\ 1 \end{pmatrix} = 1\cdot\begin{pmatrix} 1 \\ -1 \end{pmatrix} + 0\cdot\begin{pmatrix} 2 \\ 1 \end{pmatrix} + (-1)\cdot\begin{pmatrix} 3 \\ 0 \end{pmatrix} + 1\begin{pmatrix} 4 \\ 2 \end{pmatrix}$

$(2\ 1\ 1) = (-1)\cdot(1\ 0\ 1) + 1\cdot(-1\ 1\ 0) + 0\cdot(0\ 1\ -1) + 2\cdot(2\ 0\ 1)$

- Identity Matrix : $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

  $AI = A$ , $IA = A$.

- Inverse Matrix : $A \in \mathbb{R}^{n \times n}$ , $AA^{-1} = I$

  $(A^{-1})^{-1} = A$ $\qquad A^{-1}A = I$

  Ex: $\underset{A}{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}} \underset{A^{-1}}{\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

- Gaussian Elimination for finding $A^{-1}$ :

  Use row ops on $[A \mid I]$ to find its RREF

  1) If RREF is $\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & a & b & c & d \\ 0 & 1 & 0 & 0 & e & f & g & h \\ 0 & 0 & 1 & 0 & i & j & k & l \\ 0 & 0 & 0 & 1 & m & n & o & p \end{array}\right]$ ,

  then $A^{-1} = \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix}$

  2) If not enough pivots, then <u>not invertible</u>

  non singular means invertible $\qquad$ also called singular.

- A <u>square</u> linear system $A\vec{x} = \vec{b}$ , means that

  A is square.

  If $A^{-1}$ exists $\Rightarrow$ $\vec{x} = A^{-1}\vec{b}$ is the only sol.

  If many/no sols $\Rightarrow$ A is singular.

Ex: $A\vec{x} = \vec{b}$

$$\underbrace{\begin{pmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{pmatrix}}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underbrace{\begin{pmatrix} 2 \\ 8 \\ 10 \end{pmatrix}}$$

To find $A^{-1}$:

Step 0: set up $[A \mid I]$

$$\left[\begin{array}{ccc|ccc} 2 & 4 & -2 & 1 & 0 & 0 \\ 4 & 9 & -3 & 0 & 1 & 0 \\ -2 & -3 & 7 & 0 & 0 & 1 \end{array}\right]$$

Step 1: use row ops to transform it to RREF

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 27/4 & -11/4 & 3/4 \\ 0 & 1 & 0 & -11/4 & 5/4 & -1/4 \\ 0 & 0 & 1 & 3/4 & -1/4 & 1/4 \end{array}\right]$$

$$\Rightarrow A^{-1} = \frac{1}{4}\begin{bmatrix} 27 & -11 & 3 \\ -11 & 5 & -1 \\ 3 & -1 & 1 \end{bmatrix}$$

$$\Rightarrow \vec{x} = \underbrace{A^{-1}}\underbrace{\vec{b}} = \frac{1}{4}\begin{bmatrix} 27 & -11 & 3 \\ -11 & 5 & -1 \\ 3 & -1 & 1 \end{bmatrix}\underbrace{\begin{pmatrix} 2 \\ 8 \\ 10 \end{pmatrix}}$$

$$= \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$$

- Elementary Matrices: generated by one row/col operation on the identity matrix

E is always invertible. $E^{-1}$ is also an elementary matrix, generated by the inverse operation.

Type I $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, $E^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

Type II $E = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$, $E^{-1} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}$

Type III $E = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $E^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$

- $EA$ is equivalent to row op on $A$

$AE$ is equivalent to col op on $A$

Ex: $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+c & b+d \\ c & d \end{pmatrix}$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & a+b \\ c & c+d \end{pmatrix}$

If we use elementary matrices to represent the Gaussian Elimination, then every time we perform a row op, it's the same as multiplying an $E$ matrix from the left.

Assume we need $m$ row ops to get RREF of $[A|I]$

1st row op: $E_1[A \mid I] = [E_1A \mid E_1I] = [E_1A \mid E_1]$

2nd row op: $\underline{E_2}\,[E_1A \mid E_1] = [\underline{E_2E_1A} \mid \underline{E_2E_1}]$

3rd row op: $\underline{E_3}\,[E_2E_1A \mid E_2E_1] = [E_3E_2E_1A \mid E_3E_2E_1]$

$\vdots$

m-th row op: $E_m[E_{m-1}E_{m-2}\cdots E_1A \mid E_{m-1}E_{m-2}\cdots E_1]$

$$= [E_mE_{m-1}\cdots E_1A \mid E_mE_{m-1}\cdots E_1]$$

$$= [\,I \mid E_mE_{m-1}\cdots E_1]$$

① This means $\underline{E_mE_{m-1}\cdots E_1}A = I$

$$\Rightarrow A^{-1} = E_mE_{m-1}\cdots E_1$$

② We know that $\begin{cases} E_1E_1^{-1} = I \\ E_2E_2^{-1} = I \\ \vdots \\ E_mE_m^{-1} = I \end{cases}$

So $\underbrace{(E_mE_{m-1}\cdots E_1)}^{A^{-1}}\underbrace{(E_1^{-1}E_2^{-1}\cdots E_{m-1}^{-1}E_m^{-1})}$

$$= E_m E_{m-1} \cdots E_2 \underbrace{E_1 E_1^{-1}}_{I} E_2^{-1} \cdots E_{m-1}^{-1} E_m^{-1}$$

$$= E_m E_{m-1} \cdots \underbrace{E_2 E_2^{-1}}_{I} \cdots E_{m-1}^{-1} E_m^{-1}$$

$$\vdots$$

$$= E_m E_{m-1} E_{m-1}^{-1} E_m$$

$$= E_m E_m^{-1} = I$$

$\Rightarrow$ The inverse of $\underbrace{E_m E_{m-1} \cdots E_1}_{A^{-1}}$ is $E_1^{-1} E_2^{-1} \cdots E_{m-1}^{-1} E_m^{-1}$.

The inverse of $A^{-1}$ is $A$

The inverse of $A^{-1}$ is also $A$, and inverse matrix is unique $\Rightarrow A = E_1^{-1} E_2^{-1} \cdots E_{m-1}^{-1} E_m^{-1}$.

<u>Fact</u>: Any invertible matrix $A$ is a product of some elementary matrices.

Example:

$$\left[\begin{array}{ccc|ccc} 2 & 4 & -2 & 1 & 0 & 0 \\ 4 & 9 & -3 & 0 & 1 & 0 \\ -2 & -3 & 7 & 0 & 0 & 1 \end{array}\right]$$

$$E_1 = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad ① \left(\tfrac{1}{2}r1 \to r1\right) \quad \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1/2 & 0 & 0 \\ 4 & 9 & -3 & 0 & 1 & 0 \\ -2 & -3 & 7 & 0 & 0 & 1 \end{array}\right]$$

$$E_2 = \begin{pmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad ② \left(-4r1 + r2 \to r2\right) \quad \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1/2 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ -2 & -3 & 7 & 0 & 0 & 1 \end{array}\right]$$

$$E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \quad ③ \left(2r1 + r3 \to r3\right) \quad \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1/2 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 1 & 5 & 1 & 0 & 1 \end{array}\right]$$

$$E_4 = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad ④ \quad (-2r_2 + r_1 \to r_1) \quad \left[\begin{array}{ccc|ccc} 1 & 0 & -3 & 9/2 & -2 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 1 & 5 & 1 & 0 & 1 \end{array}\right]$$

$$E_5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \qquad ⑤ \quad (-r_2 + r_3 \to r_3) \quad \left[\begin{array}{ccc|ccc} 1 & 0 & -3 & 9/2 & -2 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 4 & 3 & -1 & 1 \end{array}\right]$$

$$E_6 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/4 \end{pmatrix} \qquad ⑥ \quad (\tfrac{1}{4} r_3 \to r_3) \quad \left[\begin{array}{ccc|ccc} 1 & 0 & -3 & 9/2 & -2 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 3/4 & -1/4 & 1/4 \end{array}\right]$$

$$E_7 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \qquad ⑦ \quad (r_2 - r_3 \to r_3) \quad \left[\begin{array}{ccc|ccc} 1 & 0 & -3 & 9/2 & -2 & 0 \\ 0 & 1 & 0 & -11/4 & 5/4 & -1/4 \\ 0 & 0 & 1 & 3/4 & -1/4 & 1/4 \end{array}\right]$$

$$E_8 = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad ⑧ \quad (3r_3 + r_1 \to r_1) \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 27/4 & -11/4 & 3/4 \\ 0 & 1 & 0 & -11/4 & 5/4 & -1/4 \\ 0 & 0 & 1 & 3/4 & -1/4 & 1/4 \end{array}\right]$$

$$A = E_1^{-1} E_2^{-1} \cdots E_8^{-1}$$

$$A^{-1} = E_8 E_7 \cdots E_2 E_1$$

$$\boxed{\text{Fact}: \ A^{-1}, B^{-1} \text{ exist} \Rightarrow (AB)^{-1} = B^{-1}A^{-1}}$$

$$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$$

$$\begin{aligned} &(AB)(B^{-1}A^{-1}) \\ &= A\underbrace{BB^{-1}}A^{-1} \\ &= A \cdot I \cdot A^{-1} \\ &= A \cdot A^{-1} = I. \end{aligned}$$

# Chapter 3 Vector Spaces

- If a vector $\vec{v}$ is a linear combination of some vectors, then we say $\vec{v}$ is spanned by these vectors.

Example: $A\vec{x}=\vec{b}$

$$\begin{pmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 2 \\ 8 \\ 10 \end{pmatrix}$$ has one sol $\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}=\begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}=\begin{pmatrix} 2 \\ 8 \\ 10 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 2 \\ 8 \\ 10 \end{pmatrix}=x_0\begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix}+y_0\begin{pmatrix} 4 \\ 9 \\ -3 \end{pmatrix}+z_0\begin{pmatrix} -2 \\ -3 \\ 7 \end{pmatrix}$$

$\Rightarrow \vec{b}$ is spanned by cols of $A$.

- $S$ is a set of vectors, $\mathrm{Span}(S)$ denotes the set of all possible linear combination of vectors in $S$.

Example:

① $\mathrm{Span}\left\{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right\}=\left\{a\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \forall a\in\mathbb{R}\right\}$ is a line.

② $\mathrm{Span}\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}=\left\{a\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}+b\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \forall a,b\in\mathbb{R}\right\}$

③ $\mathrm{Span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}=\mathbb{R}^3$

- Definition

A Vector Space over real numbers is a set and

① addition is defined for any two elements in the set

② scalar multiplication is defined for a scalar in $\mathbb{R}$ and any element in the set.

③ the set is closed under these two operations.

meaning that

1) the sum of two element is still in the set

2) scalar multiplication is still in the set

④ elements in this set is called (abstract) vectors.

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Example: the following are all (abstract) vector spaces

elementary vectors

$\mathbb{R} = \{\text{all real numbers}\}$

$\mathbb{R}^2 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix}, \ \forall x, y \in \mathbb{R} \right\}$

$\mathbb{R}^3 = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \ \forall a, b, c \in \mathbb{R} \right\}$

$\mathbb{R}^4 = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}, \ \forall a, b, c, d \in \mathbb{R} \right\}$

$\mathbb{R}^{1\times 3} = \left\{ [x \quad y \quad z] : \forall x, y, z \in \mathbb{R} \right\}$

$\mathbb{R}^{2\times 2} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \ \forall a, b, c, d \in \mathbb{R} \right\}$

$\mathbb{R}^{m\times n}$

$P_2(x) = \{ ax^2 + bx + c : \ \forall a, b, c \in \mathbb{R} \}$

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Ex: Is $\text{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$ a Vector Space?

Yes.

Sol: $\forall \vec{u}, \vec{v} \in \text{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}, \quad \forall a \in \mathbb{R}$

$\vec{u} \in \text{Span}\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}\right\} \Rightarrow \vec{u} = s\begin{bmatrix}1\\2\\3\end{bmatrix},\ s \in \mathbb{R}.$

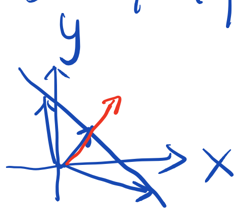
$a\vec{u} = a \cdot s\begin{bmatrix}1\\2\\3\end{bmatrix} = (as)\begin{bmatrix}1\\2\\3\end{bmatrix} \Rightarrow a\vec{u} \in \text{Span}\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}\right\}$

$\vec{v} \in \text{Span}\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}\right\} \Rightarrow \vec{v} = t\begin{bmatrix}1\\2\\3\end{bmatrix},\ t \in \mathbb{R}.$

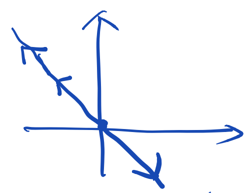
$\vec{u} + \vec{v} = (s+t)\begin{bmatrix}1\\2\\3\end{bmatrix} \in \text{Span}\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}\right\}.$

Ex: Is $\text{Span}\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$ a vector space? Yes.

Ex:




$y = -x + 1$



$y = -x$