- An abstract vector space V (such as $\mathbb{R}^2, \mathbb{R}^3, \mathbb{R}^{2\times 2}, \ldots$) is a set satisfying

  ① two operations are defined: $+$ and scalar multiplication

  ② closed under two operation: $\forall \vec{u}, \vec{v} \in V$, $\forall a \in \mathbb{R}$

  1) $\vec{u} + \vec{v} \in V$

  2) $a\vec{u} \in V$

$$a\vec{u} + b\vec{v} + c\vec{w} \in V$$

Examples:

$\mathbb{R} = \{\text{all real numbers}\}$

elementary vectors

$\mathbb{R}^2 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix}, \ \forall x, y \in \mathbb{R} \right\}$

$\mathbb{R}^3 = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \ \forall a, b, c \in \mathbb{R} \right\}$

$\mathbb{R}^4 = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}, \ \forall a, b, c, d \in \mathbb{R} \right\}$

$\mathbb{R}^{1\times 3} = \left\{ \begin{bmatrix} x & y & z \end{bmatrix} : \forall x, y, z \in \mathbb{R} \right\}$

$\mathbb{R}^{2\times 2} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \ \forall a, b, c, d \in \mathbb{R} \right\}$

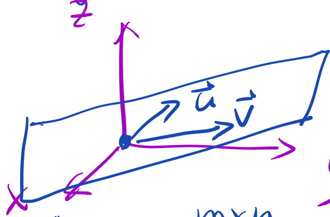
$\mathbb{R}^{m\times n}$

$P_2(\mathbb{R}) = \{ ax^2 + bx + c : \ \forall a, b, c \in \mathbb{R} \}$

- Elements in an abstract vector space are called abstract vectors

- There is always a zero vector $\vec{0}$ s.t. $\vec{v}+\vec{0}=\vec{v}$
  1) For $\mathbb{R}^n$, $\vec{0}=\begin{pmatrix}0\\0\\\vdots\\0\end{pmatrix}$
  2) For $\mathbb{R}^{m\times n}$, $\vec{0}$ is the zero matrix of size $m\times n$.
  3) For $P_2(\mathbb{R})$, $\vec{0}$ is the zero polynomial $p(x)=0$.
- Theorem: 1) $0\cdot\vec{v}=\vec{0}$, $\forall\vec{v}\in V$
  2) Closedness $\Rightarrow$ $\vec{0}=0\cdot\vec{v}\in V$.
- Definition: If V is a vector space, $W\subseteq V$, (is a subset of)
  and W is also a vector space, then W is called a subspace of V.
- Example: ① $\text{Span}\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}\right\}=\left\{a\begin{bmatrix}1\\2\\3\end{bmatrix},\ \forall a\in\mathbb{R}\right\}$ is a subspace of $\mathbb{R}^3$.
  ② $\text{Span}\left\{\begin{bmatrix}1\\0\\1\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$ is a subspace of $\mathbb{R}^3$.
- Theorem: $\forall S\subseteq V$, $\text{Span}(S)$ is a subspace of V.
  $\text{Span}(S)\subseteq V$
- If $\boxed{\vec{0}\notin W}$, then W is not a subspace.
  Example: any plane that does not pass the origin cannot be a subspace of $\mathbb{R}^3$.

Ex: Is any plane passing the origin a subspace?

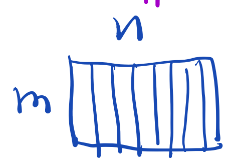


$= \text{Span}\{\vec{u}, \vec{v}\}$

- Definition. $A \in \mathbb{R}^{m \times n}$

① Span{all cols of A} is called the column space of A, denoted as $\text{Col}(A) \subseteq \mathbb{R}^m$

② Span{all rows of A} is called the row space of A, denoted as $\text{Row}(A) \subseteq \mathbb{R}^{1 \times n}$



Example:
$$\begin{pmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \\ 10 \end{pmatrix}$$
$$A \quad \vec{x} \quad = \quad \vec{b}$$

$$\text{Col}(A) = \left\{ a\begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} + b\begin{bmatrix} 4 \\ 9 \\ -3 \end{bmatrix} + c\begin{bmatrix} -2 \\ -3 \\ 7 \end{bmatrix}, \ \forall a, b, c \in \mathbb{R} \right\}$$

$\text{Row}(A) =$
$$\{ a[2 \ \ 4 \ \ -2] + b[4 \ \ 9 \ \ -3] + c[-2 \ \ -3 \ \ 7], \ \forall a, b, c \in \mathbb{R} \}$$

- $A\vec{x} = \vec{b}$ has at least one sol if and only if $\vec{b} \in \text{Col}(A)$

Proof: "if" Assume $\vec{b} \in Col(A)$, then there are $a_0, b_0, c_0 \in \mathbb{R}$ s.t.

$$\vec{b} = a_0 \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} + b_0 \begin{bmatrix} 4 \\ 9 \\ -3 \end{bmatrix} + c_0 \begin{bmatrix} -2 \\ -3 \\ 7 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} \begin{bmatrix} a_0 \\ b_0 \\ c_0 \end{bmatrix} = \vec{b}$$

$\Rightarrow \begin{bmatrix} a_0 \\ b_0 \\ c_0 \end{bmatrix}$ is a sol.

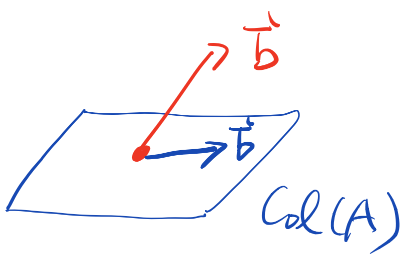
"only if" Assume $A\vec{x} = \vec{b}$ has one sol $\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$

$$\Rightarrow \begin{pmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \\ 10 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 2 \\ 8 \\ 10 \end{pmatrix} = x_0 \begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix} + y_0 \begin{pmatrix} 4 \\ 9 \\ -3 \end{pmatrix} + z_0 \begin{pmatrix} -2 \\ -3 \\ 7 \end{pmatrix}$$

$\Rightarrow \vec{b}$ is spanned by cols of A.

- For $A \in \mathbb{R}^{3\times 3}$, if $Col(A) = \mathbb{R}^3$, $A\vec{x} = \vec{b}$ always has at least one sol for any $\vec{b}$.

- 

$\vec{x} \in \mathbb{R}^n$

- Definition: all solutions to $A\vec{x} = \vec{0}$ form a subspace in $\mathbb{R}^n$, called null space of A, denoted as Null(A).

$A \in \mathbb{R}^{m\times n}$

Check closedness: $\forall \vec{u}, \vec{v} \in Null(A)$, $a \in \mathbb{R}$

$\vec{u} \in Null(A) \Rightarrow A\vec{u} = \vec{0}$
$\vec{v} \in Null(A) \Rightarrow A\vec{v} = \vec{0}$
$\Rightarrow A\vec{u} + A\vec{v} = \vec{0} + \vec{0}$
$\Rightarrow A(\vec{u}+\vec{v}) = \vec{0}$
$\Rightarrow \vec{u}+\vec{v} \in Null(A)$

$A\vec{u} = 0 \Rightarrow aA\vec{u} = a\vec{0} \Rightarrow A(a\vec{u}) = \vec{0}$
$\Rightarrow a\vec{u} \in Null(A)$.

- Example: Matrix Form $A\vec{x} = \vec{0}$

$$\begin{pmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Augmented Matrix $[A \mid \vec{0}]$

$$\left(\begin{array}{ccc|c} 2 & 4 & -2 & 0 \\ 4 & 9 & -3 & 0 \\ -2 & -3 & 7 & 0 \end{array}\right)$$

RREF is

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$\Rightarrow \vec{0}$ is the only sol

$\Rightarrow Null(A) = \left\{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right\}$

- Example: if RREF of $[A \mid \vec{0}]$ is

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \left[\begin{array}{cccc|c} 1 & \boxed{\begin{matrix} 2 \\ 0 \\ 0 \end{matrix}} & \begin{matrix} 0 \\ 1 \\ 0 \end{matrix} & \boxed{\begin{matrix} 0 \\ 0 \\ 0 \end{matrix}} & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \end{array}\right]$$

then $y=t \Rightarrow z=0$, $x=-2y=-2t$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{pmatrix} -2t \\ t \\ 0 \end{pmatrix} = t\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \quad \forall t \in \mathbb{R}$$

$$\Rightarrow \text{Null}(A) = \left\{ t\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \forall t \in \mathbb{R} \right\} = \text{Span}\left\{\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}\right\}$$

- Number of pivots in RREF of $[A \mid \vec{0}]$ plus number of free parameters in Null(A) is n. $A \in \mathbb{R}^{m\times n}$

- $A\vec{x} = \vec{0}$: homogeneous linear system

  $A\vec{x} = \vec{b}$: non homogeneous linear system

  Let $\vec{x}_p$ be a particular sol to $A\vec{x}=\vec{b}$ $\Rightarrow A\vec{x}_p = \vec{b}$

  Let $\vec{u}$ be any sol to $A\vec{x}=\vec{0}$ $\Rightarrow A\vec{u}=\vec{0}$

  $\Rightarrow \underline{A(\vec{x}_p+\vec{u}) = \vec{b}+\vec{0}}$

  $\Rightarrow \underline{\vec{x}_p+\vec{u}}$ is also a sol to $A\vec{x}=\vec{b}$

  $\Rightarrow$ All sols to $A\vec{x}=\vec{b}$ is $\vec{x}_p + \text{Null}(A)$

  $\text{Null}(A) = \{\text{all sols to } A\vec{x}=\vec{0}\}$

Example: Assume RREF of $[A \mid \vec{b}]$ is

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$\Rightarrow$ z is free

Set $z=t$, then solve for the others.

Then RREF of $[A \mid \vec{0}]$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$z=t \Rightarrow y=t$, $x=-2y-3t=-5t$

Second row $\Rightarrow$ $y - z = 1 \Rightarrow y = 1 + z = 1 + t$

First row $\Rightarrow$ $x + 2y + 3z = 0$

$$\Rightarrow x = -2y - 3z$$
$$= -2(1+t) - 3t$$
$$= -5t - 2$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -5t-2 \\ 1+t \\ t \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -5t \\ t \\ t \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + \underline{t \begin{pmatrix} -5 \\ 1 \\ 1 \end{pmatrix}},$$
$$\forall t \in \mathbb{R}.$$

- Definition (Linear Independence)

A set of (abstract) vectors $S = \{\vec{v}_1, \cdots, \vec{v}_n\}$ is called linearly dependent if there are scalars $a_1, \cdots, a_n$ which are not all zeros, s.t.
$$a_1 \vec{v}_1 + a_2 \vec{v}_2 + \cdots + a_n \vec{v}_n = \vec{0}.$$
Otherwise, S is linearly independent.

Remark: As long as one of a_i is not zero, it satisfies the definition

Example: ① $S = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ is linearly independent

$$a\begin{bmatrix}1\\0\end{bmatrix}+b\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}1&1\\0&1\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

Augmented Matrix is $\left[\begin{array}{cc|c}1&1&0\\0&1&0\end{array}\right]$

RREF is $\left[\begin{array}{cc|c}1&0&0\\0&1&0\end{array}\right]$

$\Rightarrow \begin{bmatrix}a\\b\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix} \Rightarrow$ only zero sol

$\Rightarrow$ linearly independent.

② If two vectors $\vec{u}, \vec{v}$ are parallel in $\mathbb{R}^3$, then $\{\vec{u}, \vec{v}\}$ is dependent

$\vec{u} \parallel \vec{v} \Rightarrow \vec{u} = a\vec{v}$ for some $a \in \mathbb{R}$

$\Rightarrow \vec{u} - a\vec{v} = \vec{0}$