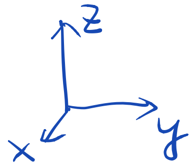
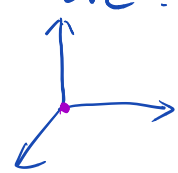
Review:

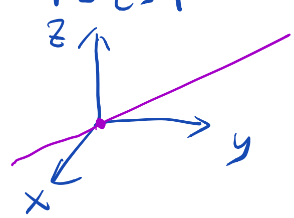
- A Vector Space $V = \mathbb{R}^3$ (or $V = \mathbb{R}^{1\times3}$) can be perceived as:



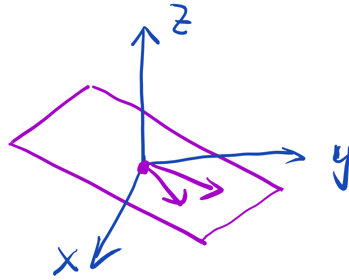
- We call W a subspace of V if $W \subseteq V$ and W is also a vector space.
- A vector space or a subspace is closed under two operations. $\vec{0} = 0 \cdot \vec{v} \in W$
- $\vec{0}$ is always included in a subspace.
- All possible subspaces of $\mathbb{R}^3$ are:
  1) $W = \{\vec{0}\} = \left\{\begin{bmatrix}0\\0\\0\end{bmatrix}\right\}$



  2) $W = \text{Span}\left\{\begin{bmatrix}a\\b\\c\end{bmatrix}\right\}$ where $\begin{bmatrix}a\\b\\c\end{bmatrix} \neq \begin{bmatrix}0\\0\\0\end{bmatrix}$.



  3) $W = \text{Span}\left\{\begin{bmatrix}a\\b\\c\end{bmatrix}, \begin{bmatrix}d\\e\\f\end{bmatrix}\right\}$ where $\begin{bmatrix}a\\b\\c\end{bmatrix}, \begin{bmatrix}d\\e\\f\end{bmatrix}$ are not on the same line.



4) $W = \text{Span}\left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \begin{bmatrix} d \\ e \\ f \end{bmatrix}, \begin{bmatrix} g \\ h \\ i \end{bmatrix} \right\} = \mathbb{R}^3$

not on the same plane.

- The column space of a matrix A is the subspace spanned by all column vectors of A.

Example: ① $A = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}$

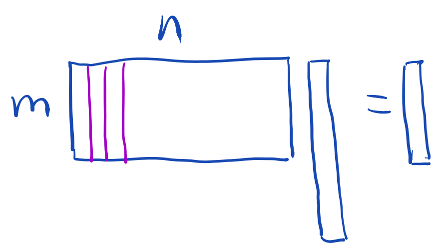
$$\text{Col}(A) = \text{Span}\left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \begin{bmatrix} d \\ e \\ f \end{bmatrix}, \begin{bmatrix} g \\ h \\ i \end{bmatrix} \right\} \subseteq \mathbb{R}^3$$

② $A = \begin{bmatrix} | & | & | \\ | & | & | \\ | & | & | \end{bmatrix} \Rightarrow \text{Col}(A) = \text{Span}\left\{ \begin{bmatrix} | \\ | \\ | \end{bmatrix} \right\}$

$$\text{Span}\left\{ \begin{bmatrix} | \\ | \\ | \end{bmatrix}, \begin{bmatrix} | \\ | \\ | \end{bmatrix}, \begin{bmatrix} | \\ | \\ | \end{bmatrix} \right\} = \text{Span}\left\{ \begin{bmatrix} | \\ | \\ | \end{bmatrix} \right\}$$

③ $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{Col}(A) = \text{Span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

- The null space of A is the set of all solutions to the homogeneous linear system $A\vec{x} = \vec{0}$ and it is a subspace of $\mathbb{R}^n$ if $A \in \mathbb{R}^{m \times n}$



$\text{Col}(A) \subseteq \mathbb{R}^m$

$\text{Null}(A) \subseteq \mathbb{R}^n$

$$A \cdot \vec{x} = \vec{b}$$

Example: ① $A = \begin{pmatrix} 1 & 3 & 2 \\ 1 & 2 & 3 \\ 0 & 1 & -1 \end{pmatrix}$ Null(A) $= \{$all sols to $A\vec{x}=\vec{0}\}$

The RREF of $[A \mid \vec{0}]$ is $\begin{pmatrix} 1 & 3 & 2 \\ 1 & 2 & 3 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} 1 & 0 & \boxed{5} & 0 \\ 0 & 1 & \boxed{-1} & 0 \\ 0 & 0 & \boxed{0} & 0 \end{array}\right)$$

$$z = t \Rightarrow \begin{cases} y = z = t \\ x = -5z = -5t \end{cases}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -5t \\ t \\ t \end{pmatrix} = t\begin{pmatrix} -5 \\ 1 \\ 1 \end{pmatrix}, \quad \forall t \in \mathbb{R}.$$

$$\Rightarrow \text{Null}(A) = \text{Span}\left\{\begin{bmatrix} -5 \\ 1 \\ 1 \end{bmatrix}\right\}$$

② All sols to $A\vec{x} = \vec{b}$ for

$$\left(\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 1 & 2 & 3 & 0 \\ 0 & 1 & -1 & 1 \end{array}\right)$$

RREF of $[A \mid \vec{b}]$ is

$$\left(\begin{array}{ccc|c} 1 & 0 & 5 & -2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$z = t \Rightarrow \begin{cases} y = 1 + z = 1 + t \\ x = -2 - 5z = -2 - 5t \end{cases}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2-5t \\ 1+t \\ t \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -5t \\ t \\ t \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + \underbrace{t\begin{pmatrix} -5 \\ 1 \\ 1 \end{pmatrix}}_{\text{Null}(A)} \quad \forall t \in \mathbb{R}.$$

The set of sols to $A\vec{x}=\vec{b}$ is equal to $\begin{pmatrix}-2\\1\\0\end{pmatrix} + \underbrace{\mathrm{Span}\left\{\begin{bmatrix}-5\\1\\1\end{bmatrix}\right\}}_{\mathrm{Null}(A)}$

- Fact: all sols to $A\vec{x}=\vec{b}$ can be written as

$$\vec{x}_p + \mathrm{Null}(A)$$

where $\vec{x}_p$ is a sol to $A\vec{x}=\vec{b}$.

- Definition (Linear Independence)

A set of (abstract) vectors $S=\{\vec{v}_1, \dots, \vec{v}_n\}$ is called linearly dependent if there are scalars $a_1, \dots, a_n$ which are not all zeros, s.t.

$$a_1\vec{v}_1 + a_2\vec{v}_2 + \dots + a_n\vec{v}_n = \vec{0}.$$

Otherwise, S is linearly independent.

Remark: As long as one of a_i is not zero, it satisfies the definition

Examples: Assume $2\begin{bmatrix}a\\b\\c\end{bmatrix} + \begin{bmatrix}d\\e\\f\end{bmatrix} - \begin{bmatrix}g\\h\\i\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$,

then they are linearly dependent, and

$$\begin{bmatrix}a\\b\\c\end{bmatrix} = -\frac{1}{2}\begin{bmatrix}d\\e\\f\end{bmatrix} + \frac{1}{2}\begin{bmatrix}g\\h\\i\end{bmatrix} \in \mathrm{Span}\left\{\begin{bmatrix}d\\e\\f\end{bmatrix}, \begin{bmatrix}g\\h\\i\end{bmatrix}\right\}$$

$$\Rightarrow \operatorname{Span}\left\{\begin{bmatrix} a \\ b \\ c \end{bmatrix}, \begin{bmatrix} d \\ e \\ f \end{bmatrix}, \begin{bmatrix} g \\ h \\ i \end{bmatrix}\right\} = \operatorname{Span}\left\{\begin{bmatrix} d \\ e \\ f \end{bmatrix}, \begin{bmatrix} g \\ h \\ i \end{bmatrix}\right\}$$

So dependency means redundancy: at least one vector can be spanned by others.

- Two vectors in $\mathbb{R}^3$ are dependent means they are on the same line:

  assume $\vec{u}, \vec{v} \in \mathbb{R}^3$ are dependent

  then there are numbers a, b s.t.

  $$a\vec{u} + b\vec{v} = \vec{0}$$

  and at least one of a, b is not 0.

  With loss of generality, let $a \neq 0$, then

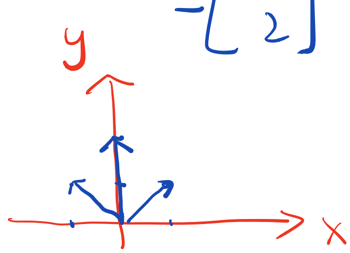
  $$a\vec{u} = -b\vec{v}$$

  $$\Rightarrow \vec{u} = -\frac{b}{a}\vec{v}$$

  $$\Rightarrow \vec{u} \parallel \vec{v}$$

- Example: $\left\{\begin{bmatrix} 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$ are dependent

  because $-\begin{bmatrix} 0 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$



Examples: $S=\{\vec{0}\}$ is dependent.

① If S contains $\vec{0}$, then S is dependent.

$S=\{\vec{0},\vec{u},\vec{v},\vec{w}\} \Rightarrow 2\cdot\vec{0}+0\cdot\vec{u}+0\cdot\vec{v}+0\cdot\vec{w}=\vec{0}$

② Check whether cols of $A=\begin{bmatrix}1&0&3\\2&1&5\\1&0&3\end{bmatrix}$ are independent: assume $\underline{a\begin{bmatrix}1\\2\\1\end{bmatrix}+b\begin{bmatrix}0\\1\\0\end{bmatrix}+c\begin{bmatrix}3\\5\\3\end{bmatrix}}=\underline{\begin{bmatrix}0\\0\\0\end{bmatrix}}$

then $\underline{\begin{bmatrix}1&0&3\\2&1&5\\1&0&3\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}}$

$\Rightarrow \left[\begin{array}{ccc|c}1&0&3&0\\2&1&5&0\\1&0&3&0\end{array}\right]$

$\Rightarrow \left[\begin{array}{ccc|c}1&0&3&0\\0&1&-1&0\\0&0&0&0\end{array}\right]$

there are nonzero solutions

$\Rightarrow$ dependent.

③ Check whether rows of A are dependent:

$a\begin{bmatrix}1&0&3\end{bmatrix}+b\begin{bmatrix}2&1&5\end{bmatrix}+c\begin{bmatrix}1&0&3\end{bmatrix}=\begin{bmatrix}0&0&0\end{bmatrix}$

$$\Rightarrow \begin{cases}a+2b+c=0\\ b=0\\ 3a+5b+3c=0\end{cases}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 3 & 5 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

④ Check whether $\left\{ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$ is independent.

Sol: $a\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + c\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

equate each entry in the matrix $\Rightarrow \begin{cases} a+b=0 \\ a+c=0 \\ \boxed{0=0} \\ a+b=0 \end{cases}$

$$\Rightarrow \underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}}_{4\times 3} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

⑤ Check whether $\{1+x^2, 1-x+x^2, 1\}$ is independent

Let $a(1+x^2)+b(1-x+x^2)+c\cdot 1 = 0$

$$\Rightarrow (a+b+c) + (-b)\cdot x + (a+b)x^2 = 0$$

$$\Rightarrow \begin{cases} a+b=0 \\ -b=0 \\ a+b+c=0 \end{cases}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \Rightarrow \text{only zero sol}$$

$\Rightarrow$ independent.

- Definition (Basis & Dimension)

If a vector space V ①is spanned by some ②linearly independent vectors, then these vectors are called a basis of V. The number of basis vectors is called the dimension of V.

Remark: For a vector space, there could be many different bases.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Examples:

① $V = \mathbb{R}^3$, $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis.

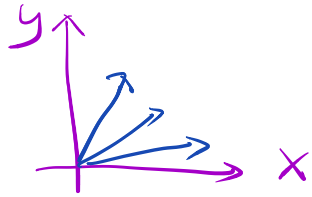
② $V = \mathbb{R}^3$, $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ is also a basis.

③ $V = \mathbb{R}^{2\times 2}$, $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is a basis

$ax^2+bx+c = a\cdot x^2 + b\cdot x + c\cdot 1$

④ $V = P_2(\mathbb{R}) = \text{Span}\{1, x, x^2\}$

$\mathbb{R}^2$:



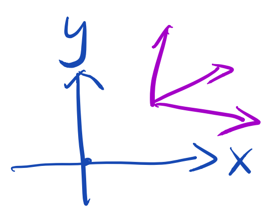
Theorem For any vector space V,

① different bases have the same number of vectors

② the maximum number of linearly independent vectors is the dimension of V.

③ the minimum number of the vectors spanning V is the dimension of V.

Example: ① $V = \mathbb{R}^2$



$$V = \operatorname{Span}\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\} \Rightarrow \dim(V) = 2.$$

Three vectors in $\mathbb{R}^2$ are always dependent.

② $V = \mathbb{R}^3$

$$x\begin{bmatrix} a \\ b \\ c \end{bmatrix} + y\begin{bmatrix} d \\ e \\ f \end{bmatrix} + z\begin{bmatrix} g \\ h \\ i \end{bmatrix} + w\begin{bmatrix} j \\ k \\ \ell \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} a & d & g & j \\ b & e & h & k \\ c & f & i & \ell \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} a & d & g & j & 0 \\ b & e & h & k & 0 \\ c & f & i & \ell & 0 \end{array}\right]$$

Three rows $\Rightarrow$ at most 3 pivots in RREF

$\Rightarrow$ at least one col without pivot

$\Rightarrow$ at least one free parameter

$\Rightarrow$ nonzero sol for $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$

$\Rightarrow$ dependent