HW #3 Solutions:

$\boxed{1(a)}$ "IF" $A = E_1 E_2 \cdots E_n \Rightarrow A^{-1} = \underline{E_n^{-1} E_{n-1}^{-1} \cdots E_1^{-1}}$

"ONLY IF" (Sep 7 Lecture Notes)

Assume we need m row ops to get RREF of $[A|I]$

1st row op: $E_1 [A \mid I] = [E_1 A \mid E_1 I] = [E_1 A \mid E_1]$

2nd row op: $\underline{E_2} [E_1 A \mid E_1] = [\underline{E_2 E_1 A} \mid \underline{E_2 E_1}]$

⋮

m-th row op: $E_m \; E_{m-1} E_{m-2} \cdots E_1 A \mid E_{m-1} E_{m-2} \cdots E_1]$

$= [E_m E_{m-1} \cdots E_1 A \mid E_m E_{m-1} \cdots E_1]$

$= [I \mid E_m E_{m-1} \cdots E_1]$

① This means $\underbrace{E_m E_{m-1} \cdots E_1}A = I$

$\Rightarrow A^{-1} = E_m E_{m-1} \cdots E_1$

② $\underbrace{(E_m E_{m-1} \cdots E_1)}_{A^{-1}} \underbrace{(E_1^{-1} E_2^{-1} \cdots E_{m-1}^{-1} E_m^{-1})}_{A} = I$

The inverse of A^{-1} is also A, and inverse matrix is unique $\Rightarrow A = E_1^{-1} E_2^{-1} \cdots E_{m-1}^{-1} E_m^{-1}$.

$\boxed{1(d)}$ (Sep 9 Lecture Notes)

Consider $\begin{pmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \\ 10 \end{pmatrix}$ as an example of $A\vec{x} = \vec{b}$

$A\vec{x} = \vec{b}$ has at least one sol if and only if $\vec{b} \in Col(A)$

Proof: "if" Assume $\vec{b} \in Col(A)$, then there is a set of coefs $a_0, b_0, c_0 \in \mathbb{R}$ s.t.

there could be many such sets
we only know there is at least one set of coefs

$$\vec{b} = a_0 \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} + b_0 \begin{bmatrix} 4 \\ 9 \\ -3 \end{bmatrix} + c_0 \begin{bmatrix} -2 \\ -3 \\ 7 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} \begin{bmatrix} a_0 \\ b_0 \\ c_0 \end{bmatrix} = \vec{b}$$

$$\Rightarrow \begin{bmatrix} a_0 \\ b_0 \\ c_0 \end{bmatrix} \text{ is a sol.}$$

"only if" Assume $A\vec{x} = \vec{b}$ has one sol $\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$

$$\Rightarrow \begin{pmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \\ 10 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 2 \\ 8 \\ 10 \end{pmatrix} = x_0 \begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix} + y_0 \begin{pmatrix} 4 \\ 9 \\ -3 \end{pmatrix} + z_0 \begin{pmatrix} -2 \\ -3 \\ 7 \end{pmatrix}$$

$\Rightarrow \vec{b}$ is spanned by cols of A.

$\boxed{1(h)}$ (Sep 14 Lecture)

① $A = \begin{pmatrix} 1 & 3 & 2 \\ 1 & 2 & 3 \\ 0 & 1 & -1 \end{pmatrix}$

$Null(A) = \{\text{all sols to } A\vec{x} = \vec{0}\}$

$$\begin{pmatrix} 1 & 3 & 2 \\ 1 & 2 & 3 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow Null(A) = Span\left\{ \begin{bmatrix} -5 \\ 1 \\ 1 \end{bmatrix} \right\}$$

② All sols to $A\vec{x}=b$ for

$$\left(\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 1 & 2 & 3 & 0 \\ 0 & 1 & -1 & 1 \end{array}\right)$$

RREF of $[A|\vec{b}]$ is

$$\left(\begin{array}{ccc|c} 1 & 0 & 5 & -2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$z=t \Rightarrow \begin{cases} y = 1+z = 1+t \\ x = -2-5z = -2-5t \end{cases}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2-5t \\ 1+t \\ t \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -5t \\ t \\ t \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + \underbrace{t\begin{pmatrix} -5 \\ 1 \\ 1 \end{pmatrix}}_{\text{Null}(A)} \quad \forall t \in \mathbb{R}$$

The set of sols to $A\vec{x}=\vec{b}$ is equal to $\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + \underbrace{\text{Span}\left\{\begin{bmatrix} -5 \\ 1 \\ 1 \end{bmatrix}\right\}}_{\text{Null}(A)}$

- Fact: ① If $\vec{b} \notin \text{Col}(A)$, $A\vec{x}=\vec{b}$ has no sols
  ② If $\vec{b} \in \text{Col}(A)$, then $A\vec{x}=\vec{b}$ has at one sol $\vec{x}_p$, and
  all sols to $A\vec{x}=\vec{b}$ can be written as

$$\vec{x}_p + \text{Null}(A)$$

$\boxed{4 \,\&\, 5}$ $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ $A^T = \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}$

Two formulae for transpose:

① $(AB)^T = B^T A^T$

$(ABC)^T = C^T B^T A^T$

A, B, C do not have to be square matrices

it is valid as long as sizes match.

② $(A^T)^{-1} = (A^{-1})^T$ (if A is invertible)

Idea of Proof (Not Required):

A is invertible $\Rightarrow A = E_1 E_2 \cdots E_n$

① Easily verify $(E^T)^{-1} = (E^{-1})^T$

② $A^T = (E_1 \cdots E_n)^T = E_n^T \cdots E_1^T$

$A^{-1} = E_n^{-1} \cdots E_1^{-1}$ $(A^{-1})^T = (E_1^{-1})^T \cdots (E_n^{-1})^T$

$\boxed{3}$ (a) $$a\begin{pmatrix}0\\-2\\-4\end{pmatrix} + b\begin{pmatrix}2\\3\\4\end{pmatrix} + c\begin{pmatrix}4\\1\\2\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\underset{A}{\begin{pmatrix}0 & 2 & 4\\-2 & 3 & 1\\-4 & 4 & 2\end{pmatrix}}\begin{pmatrix}a\\b\\c\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

(b) $$a\,(0\quad 2\quad 4) + b\,(-2\quad 3\quad 1) + c\,(-4\quad 4\quad 2) = (0\quad 0\quad 0)$$

$$\underline{(a\quad b\quad c)}\ \underline{\begin{pmatrix}0 & 2 & 4\\-2 & 3 & 1\\-4 & 4 & 2\end{pmatrix}} = (0\quad 0\quad 0)$$

$(AB)^T = B^T A^T \Rightarrow \underset{A^T}{\begin{pmatrix} 0 & -2 & -4 \\ 2 & 3 & 4 \\ 4 & 1 & 2 \end{pmatrix}} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Meaning of AB (Sep 7 lecture notes)

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ -1 & 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 1 & -1 \\ 2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 7 & 5 & 2 \\ 2 & 1 & 1 \end{pmatrix}$$

Each col in AB is a linear combination of cols in A.

Each row in AB is a linear combination of rows in B.

Review

$a\begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} + b\begin{bmatrix} 4 \\ 9 \\ -3 \end{bmatrix} + c\begin{bmatrix} -2 \\ -3 \\ 7 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

- $A = \begin{pmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{pmatrix}$, we should/can solve $A\vec{x} = \vec{0}$ for all the following questions:

① Solve the homogeneous linear system $A\vec{x} = \vec{0}$

② Find $\text{Null}(A) = \{\text{all sols to } A\vec{x} = \vec{0}\}$

③ Determine linear independence of cols of A

④ Determine whether A is invertible

A^{-1} exists $\Rightarrow A\vec{x} = \vec{0}$ has only zero sol

So if we have nonzero sol, then A is singular (not invertible)

If only zero sol, how do we know A^{-1} should exist?

$$RREF[A|\vec{0}] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\Rightarrow \underline{E_n \cdots E_1}[A|\vec{0}] = [I|\vec{0}]$$

$$\Rightarrow [\underline{E_n \cdots E_1 A} | E_n \cdots E_1 \vec{0}]$$

$$\Rightarrow \boxed{E_n \cdots E_1} A = I$$

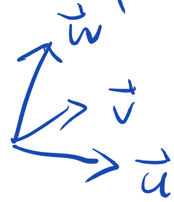
- Linear Independence:

① If some vectors $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$ are linearly dependent, then there are coefs $a_1, \cdots, a_n$ (at least one is not 0) s.t. $a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n = \vec{0}$

If $a_1 \neq 0$, then $\vec{v}_1 = -\frac{a_2}{a_1}\vec{v}_2 - \cdots - \frac{a_n}{a_1}\vec{v}_n$.

Dependence $\Rightarrow$ one vector can be spanned the others

② In $V = \mathbb{R}^3$, three vectors $\vec{u}, \vec{v}, \vec{w}$ are independent if they are not on the same plane.



- A basis for a vector space V: some $\underline{\text{independent}}_{①}$ vectors which $\underline{\text{can span } V}_{②}$.

Standard/Natural Basis:

① $V = \mathbb{R}^3$, $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

② $V = \mathbb{R}^4$, $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

③ $V = \mathbb{R}^{2\times 2}$, $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

④ $V = P_2(\mathbb{R})$, $\{1, x, x^2\}$

- There could be many different bases but they all have $\underbrace{\text{the same number}}_{\text{defined as the dimension of } V.}$ of vectors.

Example: $V = \mathbb{R}^2$, any two vectors not on the same line form a basis.

- If $\dim(V) = d$, then

① a set of independent vectors in V can have at most d vectors.

② a set of vectors spanning V have at least d vectors.

③ More than d vectors in V are always dependent.

Example: $\begin{cases} -2x+3y+z-w=0 \\ -2x+4y+3z+w=0 \end{cases}$ The sol set is 2-dimensional

$$\begin{pmatrix} -2 & 3 & 1 & -1 \\ -2 & 4 & 3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} -2 & 3 & 1 & -1 & 0 \\ -2 & 4 & 3 & 1 & 0 \end{array}\right)$$

RREF is

$$\left(\begin{array}{cccc|c} 1 & 0 & \frac{5}{2} & \frac{7}{2} & 0 \\ 0 & 1 & 2 & 2 & 0 \end{array}\right)$$

$z=s$, $w=t$

$\Rightarrow \begin{cases} y=-2s-2t \\ x=-\frac{5}{2}s-\frac{7}{2}t \end{cases}$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -\frac{5}{2}s-\frac{7}{2}t \\ -2s-2t \\ s \\ t \end{pmatrix} = \begin{pmatrix} -5/2\, s \\ -2s \\ s \\ 0 \end{pmatrix} + \begin{pmatrix} -7/2\, t \\ -2t \\ 0 \\ t \end{pmatrix}$$

$$= s\begin{pmatrix} -5/2 \\ -2 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -7/2 \\ -2 \\ 0 \\ 1 \end{pmatrix}, \quad \forall s,t \in \mathbb{R}$$

$\Rightarrow$ Sol Set is $\text{Span}\left\{\begin{bmatrix}-5/2\\-2\\1\\0\end{bmatrix}, \begin{bmatrix}-7/2\\-2\\0\\1\end{bmatrix}\right\} \subseteq \mathbb{R}^4$

① Span is a subspace

② This subspace has a basis $\left\{\begin{bmatrix}-5/2\\-2\\1\\0\end{bmatrix}, \begin{bmatrix}-7/2\\-2\\0\\1\end{bmatrix}\right\}$ thus it's 2-dim.

• Theorem: If $S = \{\vec{v}_1, \cdots, \vec{v}_n\}$ is a basis of V

then any $\vec{u} \in V$ can be written as

a $\boxed{\text{unique}}$ linear combination of $\vec{v}_1, \cdots, \vec{v}_n$

Proof for $\mathbb{R}^3$: Assume $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis of $\mathbb{R}^3$.

Then $\text{Span}(S) = \mathbb{R}^3 \Rightarrow$

$\forall \vec{u} \in \mathbb{R}^3$, $\vec{u} \in \text{Span}(S)$

Suppose there are two linear combinations:

$$\vec{u} = a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3$$
$$\vec{u} = b_1\vec{v}_1 + b_2\vec{v}_2 + b_3\vec{v}_3$$

$$\Rightarrow \vec{0} = (a_1 - b_1)\vec{v}_1 + (a_2 - b_2)\vec{v}_2 + (a_3 - b_3)\vec{v}_3$$

Independence of $S \Rightarrow \begin{cases} a_1 - b_1 = 0 \\ a_2 - b_2 = 0 \\ a_3 - b_3 = 0 \end{cases}$

$\Rightarrow$ Uniqueness

Def For $A \in \mathbb{R}^{m \times n}$,

① $\dim(\mathrm{Col}(A))$ is called col rank of A.

② $\dim(\mathrm{Row}(A))$ is called row rank of A.

③ $\dim(\mathrm{Null}(A))$ is called nullity of A.

Theorem: ① Number of pivots in $RREF(A)$ or $RREF([A \mid \vec{0}])$ is equal to col rank of A

② Number of pivots in $RREF(A)$ or $RREF([A \mid \vec{0}])$ is equal to row rank of A

$\boxed{\text{Def}}$ The rank of A is defined as number of pivots in $RREF(A)$.

Jutification on an Example:

$$A = \begin{pmatrix} 1 & 3 & 2 \\ 1 & 2 & 3 \\ 0 & 1 & -1 \end{pmatrix}$$

The RREF of $[A \mid \vec{0}]$:

$$\left(\begin{array}{ccc|c} 1 & 3 & 2 & 0 \\ 1 & 2 & 3 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right)$$

$$E_1 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (r2 - r1 \to r1) \left(\begin{array}{ccc|c} 1 & 3 & 2 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right)$$

$$E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (-r2 \to r2) \left(\begin{array}{ccc|c} 1 & 3 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right)$$

$$E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \qquad (r3 - r2 \to r3) \left(\begin{array}{ccc|c} 1 & 3 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$E_4 = \begin{pmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (r1 - 3r2 \to r1) \left(\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\Rightarrow \quad E_4 E_3 E_2 E_1 \begin{pmatrix} 1 & 3 & 2 \\ 1 & 2 & 3 \\ 0 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Downarrow$$

$$\begin{pmatrix} 1 & 3 & 2 \\ 1 & 2 & 3 \\ 0 & 1 & -1 \end{pmatrix} = \underbrace{E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1}}_{B} \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 3 & 2 \\ 1 & 2 & 3 \\ 0 & 1 & -1 \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow$ each row in the left hand side is a linear combination of $[1\ 0\ 5]$, $[0\ 1\ -1]$ and $[0\ 0\ 0]$.

$\Rightarrow$ $\text{Row}(A) = \text{Span}\{[1\ 0\ 5], [0\ 1\ -1]\}$

Col Rank: $\text{RREF}[A \mid \vec{0}]$

$$= \left(\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$\Rightarrow$ $A\vec{x} = \vec{0}$ has nonzero sols.

$\Rightarrow$ Cols of A are dependent.

$$\underbrace{E_4 E_3 E_2 E_1}_{C} \underbrace{\begin{pmatrix} \boxed{\begin{matrix}1\\1\\0\end{matrix}} & \boxed{\begin{matrix}3\\2\\1\end{matrix}} & \begin{matrix}2\\3\\-1\end{matrix} \end{pmatrix}}_{A} = \begin{pmatrix} \boxed{\begin{matrix}1\\0\\0\end{matrix}} & \boxed{\begin{matrix}0\\1\\0\end{matrix}} & \begin{matrix}5\\-1\\0\end{matrix} \end{pmatrix}$$

$$\Rightarrow E_4 E_3 E_2 E_1 \begin{pmatrix} 1 & 3 \\ 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$a\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{pmatrix} 1 & 3 \\ 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad E_4 E_3 E_2 E_1 \begin{pmatrix} 1 & 3 \\ 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = E_4 E_3 E_2 E_1 \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$