Review

Def For $A \in \mathbb{R}^{m \times n}$,

① $\dim(\mathrm{Col}(A))$ is called col rank of A

② $\dim(\mathrm{Row}(A))$ is called row rank of A

③ $\dim(\mathrm{Null}(A))$ is called nullity of A.

Theorem: ① Number of pivots in RREF(A)
or RREF($[A \mid \vec{0}]$)
is equal to col rank of A

② Number of pivots in RREF(A)
or RREF($[A \mid \vec{0}]$)
is equal to row rank of A

$\boxed{\text{Def}}$ The rank of A is defined as number of pivots in RREF(A).

Assume $A \in \mathbb{R}^{m \times n}$, let r be its rank.
Then RREF$[A \mid \vec{0}]$ has r leading ones.

How to find a basis for Null(A), Col(A), Row(A)

① For Null(A), solve $A\vec{x}=\vec{0}$ and write Null(A) as a span of $(n-r)$ vectors, which are the basis vectors.

Example: $A=[1\ \ 1\ \ 1]$

$$\text{RREF}[A|\vec{0}] = [1\ \ \boxed{1}\ \ \boxed{1}\ |\ 0]$$

$$\Rightarrow \begin{cases} y=s \\ z=t \end{cases} \Rightarrow x=-s-t$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -s-t \\ s \\ t \end{pmatrix} = \begin{pmatrix} -s \\ s \\ 0 \end{pmatrix} + \begin{pmatrix} -t \\ 0 \\ t \end{pmatrix}$$

$$= s\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \quad \forall s,t \in \mathbb{R}$$

$$\Rightarrow \text{Null}(A) = \text{Span}\left\{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}$$

$\Rightarrow \left\{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}$ is a basis for Null(A)

② Cols in A corresponding to pivots in RREF form a basis for Col (A).

Example: $A = \begin{pmatrix} 1 & 3 & 2 \\ 1 & 2 & 3 \\ 0 & 1 & -1 \end{pmatrix}$

The RREF of $[A \mid \vec{0}]$:

$$\left(\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$\Rightarrow \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \right\}$ is a basis for Col (A)

Remark: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ cannot be the basis

because they cannot span $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$.

③ Basis for Row (A) : three methods

1) rows with pivots in RREF is a basis

Example: $A = \begin{pmatrix} 1 & 3 & 2 \\ 1 & 2 & 3 \\ 0 & 1 & -1 \end{pmatrix}$

The RREF of $[A \mid \vec{0}]$:

$$\left(\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$\Rightarrow \{[1\ 0\ 5], [0\ 1\ -1]\}$ is a basis

2) the rows in A corresponding to pivots
(be careful that rows might be switched during Gaussian Elimination)

Example: $\{[1\ 3\ 2], [1\ 2\ 3]\}$ is a basis.

3) Treat rows of A as abstract vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$, then solve for linear independence: $a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_3 = \vec{0}$

I. Independent $\Rightarrow \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis.

II. Dependent $\Rightarrow$ remove the ones corresponding to free parameters, then the rest is a basis.

Notice that this method applies to any abstract vector (e.g. col vectors).

Example: $A = \begin{pmatrix} 1 & 3 & 2 \\ 1 & 2 & 3 \\ 0 & 1 & -1 \end{pmatrix}$

Basis for Row(A).

$a[1\ 3\ 2] + b[1\ 2\ 3] + c[0\ 1\ -1] = [0\ 0\ 0]$

$(AB)^T = B^T A^T$

$\Rightarrow [a\ b\ c] \begin{bmatrix} 1 & 3 & 2 \\ 1 & 2 & 3 \\ 0 & 1 & -1 \end{bmatrix} = [0\ 0\ 0]$

$\Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 3 & 2 & 1 \\ 2 & 3 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 2 & 3 & -1 & 0 \end{array}\right]$

$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

So $\{[1\ 3\ 2], [1\ 2\ 3]\}$ is a basis of Row(A).

Example: Find a basis for

$\mathrm{Span}\{1+3x+2x^2,\ 1+2x+3x^2,\ x-x^2\}$

$a(\underbrace{1+3x+2x^2}) + b(\underbrace{1+2x+3x^2}) + c(x-x^2) = 0$

$(a+b) + (3a+2b+c)x + (2a+3b-c)x^2 = 0$

$$\Rightarrow \begin{cases} a + b + 0\cdot c = 0 \\ 3a + 2b + c = 0 \\ 2a + 3b - c = 0 \end{cases}$$

$$\Rightarrow \underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 3 & 2 & 1 \\ 2 & 3 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$a\underbrace{\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}} + b\underbrace{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}} + c\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 2 & 3 & -1 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So $\{1+3x+2x^2,\ 1+2x+3x^2\}$ is a basis.

Dimension Theorem: $A \in \mathbb{R}^{m \times n}$,

$$\text{rank}(A) + \text{nullity}(A) = n$$

Proof: rank(A) is # of pivots

nullity(A) is # of free parameters.

Geometric Meaning:

Example: $A = \begin{pmatrix} 1 & 3 & 2 \\ 1 & 2 & 3 \\ 0 & 1 & -1 \end{pmatrix}$ RREF $= \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$

$$\Rightarrow \begin{cases} \text{rank}(A) = 2 \\ \text{nullity}(A) = 1 \end{cases}$$

$\Rightarrow$ Col(A) is a 2-dimensional subspace of $\mathbb{R}^3$

Row(A) is a 2-dimensional subspace of $\mathbb{R}^{1 \times 3}$

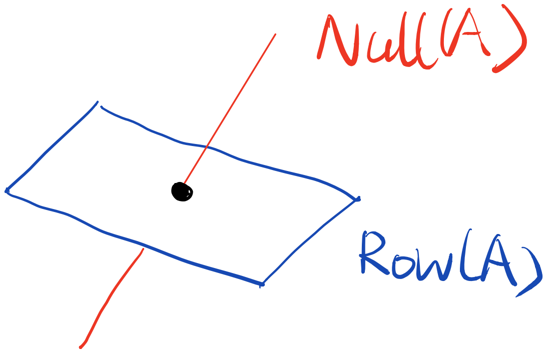
Null(A) is a 1-dimensional subspace of $\mathbb{R}^3$.

Also, $\begin{pmatrix} 1 & 3 & 2 \\ 1 & 2 & 3 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $\quad x\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + y\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + z\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$

means the dot product of rows of A with $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

are also zero.

Thus, the line Null(A) is $90°$ to the plane Row(A).



Example: $A \in \mathbb{R}^{3\times 3}$, all possibilities:

① $\text{rank}(A) = 3$, $\text{nullity}(A) = 0$

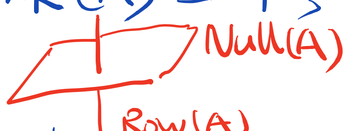
$\text{Col}(A) = \mathbb{R}^3$ $\quad$ $\text{Null}(A) = \{\vec{0}\}$

$\text{Row}(A) = \mathbb{R}^{1\times 3}$

$2 \cdot \vec{0} = \vec{0}$

② $\text{rank}(A) = 2$, $\text{nullity}(A) = 1$

③ $\text{rank}(A) = 1$, $\text{nullity}(A) = 2$



④ $\text{rank}(A) = 0$, $\text{nullity}(A) = 3$

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The following are equivalent for a square matrix $A \in \mathbb{R}^{n\times n}$ (e.g. $A \in \mathbb{R}^{3\times 3}$):

① A is invertible (nonsingular)

② The homogeneous system $A\vec{x} = \vec{0}$ has only zero sol

$$\begin{pmatrix} 1 & 3 & 2 \\ 1 & 2 & 3 \\ 0 & 1 & -1 \end{pmatrix}$$

③ Null(A) is trivial

Null(A) $= \{\vec{0}\}$

Nullity(A) $= 0$

$$a\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + c\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 & 2 \\ 1 & 2 & 3 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

④ $A\vec{x} = \vec{b}$ has a unique solution.

⑤ Cols of A are independent

Col Rank is n

Rank(A) $= n$

Col(A) $= \mathbb{R}^n$

⑥ Row of A are independent

Row Rank is n

Rank(A) $= n$ Row(A) $= \mathbb{R}^{1\times n}$

⑦ $\mathrm{RREF}(A) = I$

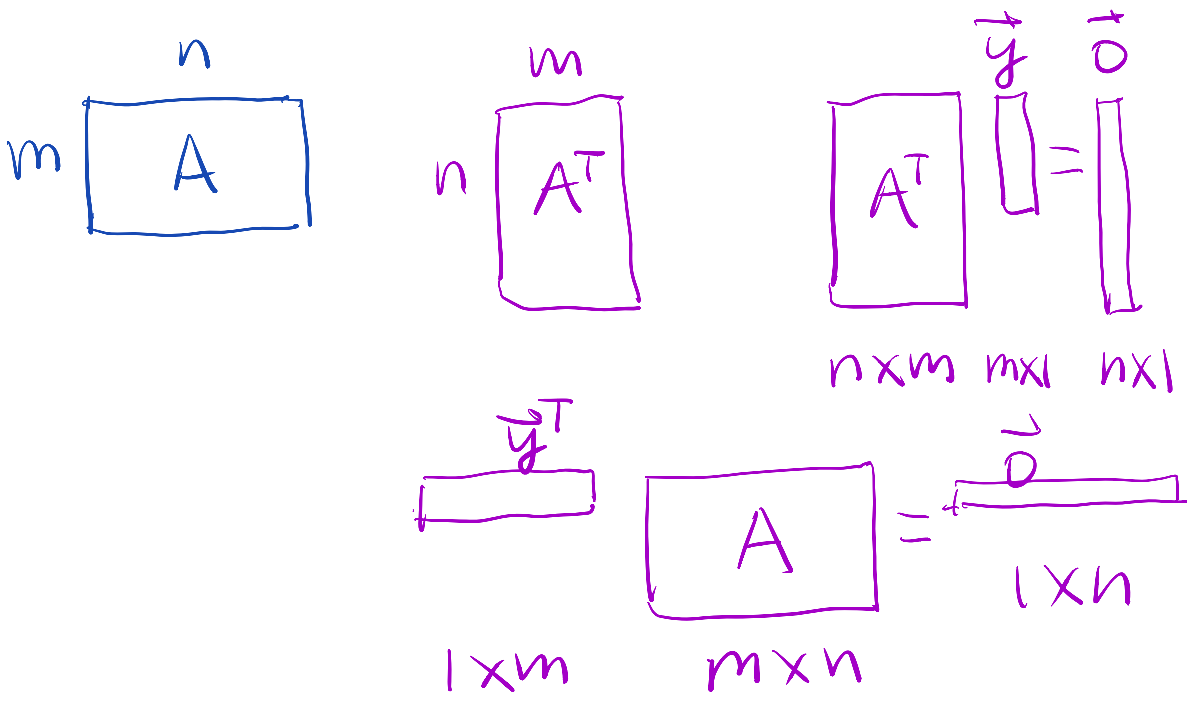
⑧ $\forall \vec{b} \in \mathbb{R}^n,\ \vec{b} \in \mathrm{Col}(A) \Rightarrow \mathbb{R}^n \subseteq \mathrm{Col}(A) \subseteq \mathbb{R}^n$

$\Rightarrow \mathrm{Col}(A) = \mathbb{R}^n$

$\boxed{\text{Def}}$ $\mathrm{Null}(A^T)$ is also called left null space of A

$$\mathrm{Null}(A^T) = \{\vec{y} \in \mathbb{R}^m : A^T \vec{y} = \vec{0}\}$$

$$\sim \{\vec{y}^T \in \mathbb{R}^{1\times m} : \vec{y}^T A = \vec{0}\}$$



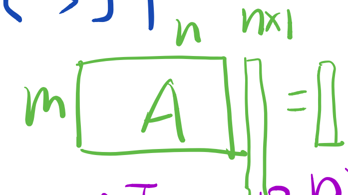
Cols of A^T correspond to rows of A

$\Rightarrow \mathrm{Col}(A^T)$ correspond to $\mathrm{Row}(A)$

Example: $A = \begin{pmatrix} 1 & 3 & 2 \\ 1 & 2 & 3 \\ 0 & 1 & -1 \end{pmatrix}$, $A^T = \begin{pmatrix} 1 & 1 & 0 \\ 3 & 2 & 1 \\ 2 & 3 & -1 \end{pmatrix}$

$$\begin{cases} \mathrm{Row}(A) = \mathrm{Span}\{[1\ 3\ 2], [1\ 2\ 3]\} \\ \mathrm{Col}(A^T) = \mathrm{Span}\left\{\begin{bmatrix}1\\3\\2\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}\right\} \end{cases}$$

We call both the row space of A.



If $A \in \mathbb{R}^{m\times n}$, $\mathrm{rank}(A) = r \Rightarrow \begin{cases} A^T \in \mathbb{R}^{n\times m} \\ \mathrm{rank}(A^T) = r \end{cases}$

Four subspaces for A.

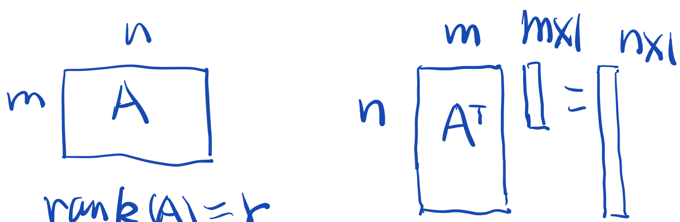
① $\mathrm{Col}(A)$ is r-dim subspace of $\mathbb{R}^m$

$\mathrm{Row}(A^T)$ is r-dim subspace of $\mathbb{R}^{1\times m}$

② $\begin{cases} \mathrm{Row}(A) \text{ is } r\text{-dim subspace of } \mathbb{R}^{1\times n} \\ \mathrm{Col}(A^T) \text{ is } r\text{-dim subspace of } \mathbb{R}^n \end{cases}$

③ $\mathrm{Null}(A)$ is $(n-r)$-dim subspace of $\mathbb{R}^n$.

④ $\mathrm{Null}(A^T)$ is $(m-r)$-dim subspace of $\mathbb{R}^m$



$\text{rank}(A) = r$

$\text{rank}(A^T) = r$

$$A^T \vec{y} = \vec{0}$$

Apply Dimension Theorem to $A^T \in \mathbb{R}^{n \times m}$

$\Rightarrow \text{rank}(A^T) + \text{Nullity}(A^T) = m$

$\Rightarrow \text{Nullity}(A^T) = m - r$

Chapter 4 Orthogonality

$\boxed{\text{Def}}$ $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $\vec{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$

The dot product of $\vec{x}$ and $\vec{y}$ is

$$\vec{x} \cdot \vec{y} = \langle \vec{x}, \vec{y} \rangle = \vec{y}^T \vec{x}$$

$$= [y_1 \ y_2 \ \cdots \ y_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

$\boxed{\text{Def}}$ We say $\vec{x}$ is orthogonal to $\vec{y}$ if

$$\langle \vec{x}, \vec{y} \rangle = 0.$$

$\boxed{\text{Def}}$ If V, W are two subspaces of the same vector space, we say V is orthogonal to W if any vector in V is orthogonal to any vector in W

Example: $A = \begin{pmatrix} 1 & 3 & 2 \\ 1 & 2 & 3 \\ 0 & 1 & -1 \end{pmatrix}$, $A^T = \begin{pmatrix} 1 & 1 & 0 \\ 3 & 2 & 1 \\ 2 & 3 & -1 \end{pmatrix}$

$$\begin{cases} Row(A) = Span\{[1 \ 3 \ 2], [1 \ 2 \ 3]\} \\ Col(A^T) = Span\left\{\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right\} \end{cases}$$

→ we call them row space of A.

$$\begin{cases} Col(A^T) \subseteq \mathbb{R}^3 \\ Null(A) \subseteq \mathbb{R}^3 \end{cases}$$

→ they are $\perp$ to each other

$$\begin{pmatrix} 1 & 3 & 2 \\ 1 & 2 & 3 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

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$$\hookrightarrow \Leftrightarrow \begin{cases} \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \\ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \\ \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \end{cases}$$