MA 351 Section 041/011
Fall 2023

Homework 1

Due on **Sep 6 before 9am** on gradescope.

To receive full credit, show necessary reasoning unless it's straightforward computation.

1. (10 pts) For a square matrix $A \in \mathbb{R}^{n\times n}$, let A_{ij} denote its entry at (i, j) position. Entries A_{ii} are called *diagonal entries* of A, e.g., $A_{11}, A_{22}, \cdots$. The sum of all diagonal entries of A is called *the trace* of A, often denoted as $tr(A)$ or $trace(A)$. Find $tr(A)$ and A_{31} for the following matrix:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 10 & 0 \\ 0 & 1 & -5 \end{pmatrix}.$$

2. (10 pts) Find the angle between the two vectors:

$$\vec{u} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \vec{v} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

3. (20 pts) For two vectors $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, verify Schwartz inequality $|\vec{u}\cdot\vec{v}| \leq \|\vec{u}\|\|\vec{v}\|$ by algebra as follows without using geometry:
   (a) Multiply out both sides of the inequality

$$(u_1v_1 + u_2v_2)^2 \leq (u_1^2 + u_2^2)(v_1^2 + v_2^2).$$

   (b) Show that the difference between those two sides equals $(u_1v_2 - u_2v_1)^2$. This cannot be negative since it is a square, thus the inequality is true.

4. (20 pts)

   Definition 1. *Multiplying two matrices.* For $A \in \mathbb{R}^{m\times n}$ and $\vec{v} \in \mathbb{R}^n$, we have defined matrix vector multiplication. Now consider another matrix $B \in \mathbb{R}^{n\times p}$ which has p columns, denoted as $\vec{b}_1, \vec{b}_2, \cdots, \vec{b}_p$. Then we can multiply A to each column of B because the sizes match: we get another

p column vectors of size m as $A\vec{b}_1, A\vec{b}_2, \cdots, A\vec{b}_p$. If we put together these vectors $A\vec{b}_1, A\vec{b}_2, \cdots, A\vec{b}_p$, we get another matrix of size $m \times p$ and define this new matrix as A multiplied to B (or product of A and B), denoted as AB. For example,

$$A = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix} \Rightarrow AB = \begin{pmatrix} 2 & 7 \\ 0 & -1 \end{pmatrix}.$$

Now compute AB for the following matrices:

(a)

$$A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 10 & 0 \\ 0 & 1 & -5 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix}.$$

(b)

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ -1 & 1 & 0 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 1 & -1 \\ 2 & 0 & 1 \end{pmatrix}.$$

5. (20 pts)Find the intersection of the three planes described by equations $2y + 4z = 2$, $-2x + 3y + z = 2$, and $-4x + 4y + 2z = 3$ respectively.
   (a) (5 pts) First obtain matrix form of the linear system and obtain the augmented matrix.

   (b) (15 pts) Use Gaussian Elimination by row operations to find the solution. You get 0 pt if solving it without using matrices.

6. (20 pts)

   Definition 2. We have defined Row Echelon Form (REF) of matrices. The Reduced Row Echelon Form (RREF) of matrices are defined as: 1) already a Row Echelon Form; 2) all leading coefficients are ones; 3) for a column containing leading ones, all other entries except the leading one are zeroes.

   Example: $A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ is in RREF with two leading ones in red, and the nonzero entries in its third column do not matter because there is no leading one in the third column.

   For the linear system of three equations $2y + 4z + w = 2$, $-2x + 3y + z - w = 2$, and $-2x + 4y + 3z + w = 3$, transform the augmented matrix to RREF by row operations.

Remark: if we just want a REF, then it is not unique since the leading coefficient can be any nonzero number. But if we want a RREF, then it is always unique.