

### Homework 3

Due on **Sep 16 before 1pm** on gradescope.

**To receive full credit, show necessary reasoning unless it's straightforward computation.**

1. (20 pts) True or false (you can simply answer T or F, no need to give justification; but try to think about why).
  - (a) A square matrix  $A$  is invertible if and only if it can be written as a product of some elementary matrices.
  - (b) A square matrix  $A$  is invertible if and only if its RREF is an identity matrix.
  - (c) A square linear system  $A\vec{x} = \vec{b}$  has a unique solution if  $A$  is invertible.
  - (d) A linear system  $A\vec{x} = \vec{b}$  has a solution if and only if  $\vec{b}$  is a linear combination of columns of  $A$ .
  - (e) For  $A \in \mathbb{R}^{m \times n}$ , its column space is a subspace of  $\mathbb{R}^n$ .
  - (f) For  $A \in \mathbb{R}^{n \times n}$ , if  $AA = I$ , then  $A$  must be either  $I$  or  $-I$ .
  - (g) For  $A \in \mathbb{R}^{m \times n}$ , its null space is a subspace of  $\mathbb{R}^m$ .
  - (h) For  $A \in \mathbb{R}^{m \times n}$ , the homogeneous system  $A\vec{x} = \vec{0}$  has as many solutions as the nonhomogeneous system  $A\vec{x} = \vec{b}$  for any  $\vec{b} \in \mathbb{R}^m$  and  $\vec{b} \neq \vec{0}$ .
  - (i) For  $A \in \mathbb{R}^{n \times n}$ , if  $\vec{b} \in Col(A)$ , all solutions to  $A\vec{x} = \vec{b}$  form a subspace of  $\mathbb{R}^n$ .
  - (j) If  $A\vec{x} = \vec{b}$  has no solutions at all, then  $\vec{b}$  cannot be a linear combi-

nation of columns of  $A$ .

2. (20 pts) For the invertible matrix

$$A = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 3 & 1 \\ -1 & 2 & 1 \end{pmatrix},$$

find suitable elementary matrices so that  $A^{-1}$  can be written as a product of them.

3. (20 pts) Let  $A = \begin{pmatrix} 0 & 2 & 4 \\ -2 & 3 & 1 \\ -4 & 4 & 2 \end{pmatrix}$ .

- (a) Determine whether columns of  $A$  are linearly independent as follows: assume there are numbers  $a, b, c$  s.t.

$$a \begin{pmatrix} 0 \\ -2 \\ -4 \end{pmatrix} + b \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} + c \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

which gives three equations about  $a, b, c$ . Solve the linear system about  $a, b, c$ . If there are nonzero solutions, then the column vectors are linearly dependent. Otherwise, they are linearly independent.

- (b) Determine whether rows of  $A$  are linearly independent as follows: assume there are numbers  $a, b, c$  s.t.

$$a(0 \ 2 \ 4) + b(-2 \ 3 \ 1) + c(-4 \ 4 \ 2) = (0 \ 0 \ 0),$$

which gives three equations about  $a, b, c$ . Solve the linear system about  $a, b, c$ . If there are nonzero solutions, then the row vectors are linearly dependent. Otherwise, they are linearly independent.

4. (20 pts)

**Definition 1** (Transpose matrix). For a matrix  $A$  of size  $m \times n$ , its transpose matrix  $A^T$  has size  $n \times m$ , and the  $j$ -th column of  $A^T$  is obtained by converting the  $j$ -th row of  $A$  to a column. For example,

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}, \quad A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}.$$

For a square matrix  $A$ ,  $A^T$  can also be viewed as flipping non-diagonal entries with respect to the diagonal entries. For example:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad A^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

(a) Find the transpose for  $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$  and  $B = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \\ j & k & l \end{pmatrix}$ .

(b) The transpose for product of matrices is similar to inverse [recall that we have a formula  $(AB)^{-1} = B^{-1}A^{-1}$  ]:

$$(AB)^T = B^T A^T. \quad (1)$$

Verify the formula (1) for the following specific matrices:

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

In other words, compute  $(AB)^T$  and  $B^T A^T$  to show they are the same.

5. (20 pts) If a square matrix  $A$  is invertible, then its transpose  $A^T$  is also invertible and the inverse of  $A^T$  is equal to the transpose of  $A^{-1}$ :

$$(A^T)^{-1} = (A^{-1})^T.$$

We can verify this formula on a specific matrix

$$A = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 3 & 1 \\ -1 & 2 & 1 \end{pmatrix}$$

as follows:

(a) Find  $A^T$  and  $A^{-1}$ .

(b) Compute the product of  $A^T$  and  $(A^{-1})^T$  to that show it is identity.